GEOMETRIC SINGULARITIES UNDER THE GIGLI-MANTEGAZZA FLOW

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Abstract. This is a report on joint work with Lashi Bandara and Michael Munn. We discuss some smoothing results for metric measure spaces. Mainly, we present regularity results for a flow of metric-measure spaces due to Gigli-Mantegazza in the context of geometric singularities modeled using rough metrics.

1. Motivation

A fundamental question in the crossroad of metric geometry and geometric analysis is whether a metric and measure space can be smoothed out i.e. given a metric and measure space (X, d, m), can one find a sequence of Riemannian (or Finslerian) manifolds (M_n, g_n) such that the corresponding metric and measure spaces converge to (X, d, m) in the measured Gromov Hausdorff distance?

There are two general ways to approach this problem. One is the direct method namely, what metric geometric conditions can one impose on X which guarantee the smoothability of X. As an example of the direct method, in [KL], Kittabeppu and myself showed if $\mathcal{R}_1 \neq \varnothing$ in a compact RCD space X, then X is $1-$dimensional hence, a Ricci limit space (once proven to be $1D$, being a Ricci limit space follows from the work of Cheeger and Colding).

The second method to approach the smoothability question is to use flow methods. In this method one needs to consider a flow with good homogenizing and smoothing properties and the hope is to be able to flow the metric and measure space X in a continuous way into a Riemannian manifold.

As an example, when X is a compact Alexandrov surface (proven by Alexandrov himself and also recently a proof given by Richard [RT] using Ricci flow techniques). Also when X is a compact $3-$dimensional polyhedron, Lebedeva-Matveev-Petrunin-Chevchishin [LMPS] showed that X is smoothable using Ricci flow.

The following are some weak formulations of Ricci flow or flows similar to the Ricci flow which can be applied to metric and measure spaces.

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- McCann and Topping outlined a notion of weak Ricci flow heuristically via contraction properties of the heat flow.
- Gigli and Mantegazza introduced a flow using the heat flow in the Wassestein space of the probability measures. The GM flow is, in a sense, a linearization of the Ricci flow for metric-measure spaces. GM flow is easier to use for smoothing purposes. Examples of smoothing via GM flow: Erbar-Juillet showed smoothing for Heisenberg group and nonsmoothing for some open Euclidean cones.
- Other weak formulations of (super) Ricci flow for metric spaces are given by Sturm (using dynamical optimal transport) and Haslhofer-Naber (using stochastic methods).

2. Smoothing properties of Gigli-Mantegazza flow

Let $\mathcal{M}$ be a smooth compact manifold, and g a smooth metric. Let $\rho^{g}_t : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ be the heat kernel of the Laplacian Δ_{g}. Fix $t > 0$, $x \in \mathcal{M}$ and $v \in \mathrm{T}_x\mathcal{M}$. Let $\varphi_{t,x,v}$ be a solution to the continuity equation

$$\text{(CE)} \qquad \begin{aligned} &-\operatorname{div}_{g}(\rho^{g}_t(x,y)\nabla\varphi_{t,x,v})(y) = (\mathrm{d}_x\rho^{g}_t(x,y))(v) \\ &\int_{\mathcal{M}} \varphi_{t,x,v}(y)\ d\mu_{g}(y) = 0. \end{aligned}$$

Then, the Gigli-Mantegazza flow g_t is a metric evolving in time by:

$$\text{(GM)} \qquad \begin{aligned} g_t(u,v)(x) &= \int_{\mathcal{M}} g(\nabla\varphi_{t,x,u}(y), \nabla\varphi_{t,x,v}(y))\ \rho^{g}_t(x,y)\ d\mu_{g}(y) \\ &= \langle \rho^{g}_t(x,\cdot)\nabla\varphi_{t,x,u}, \nabla\varphi_{t,x,v}\rangle_{L^2(\mathcal{M},g)}. \end{aligned}$$

The GM flow is tangent to the Ricci flow in the following way. Let $\gamma : [0,1] \to \mathcal{M}$ be a g-geodesic. Then,

$$\partial_t g_t(\dot\gamma(s), \dot\gamma(s))\big|_{t=0} = -2\mathrm{Ric}_{g}(\dot\gamma(s), \dot\gamma(s)),$$

That is, the metrics $t \mapsto g_t$ is *tangential* to the Ricci flow almost-everywhere along g-geodesics. Main redeeming feature: this can be defined rather easily as a flow of distance metrics d_t for metric spaces $(\mathcal{X}, \mathrm{d}, \mu)$ that satisfy the *Riemannian Curvature Dimension* (RCD) condition.

2.1. Wasserstein space and synthetic Ricci curvature.

Let $(\mathcal{X}, \mathrm{d}, \mu)$ be a compact measure metric geodesic space. Denote set of probability measures by $\mathscr{P}(\mathcal{X})$. For $\nu, \sigma \in \mathscr{P}(\mathcal{X})$, a *transport plan* between ν and σ is measure π on $\mathcal{X} \times \mathcal{X}$ such that

$$\pi(A \times \mathcal{X}) = \nu(A) \quad \text{and} \quad \pi(\mathcal{X} \times B) = \sigma(B).$$

Define:

$$W_2(\nu,\sigma)^2 = \inf\left\{\int_{\mathcal{X}\times\mathcal{X}} \mathrm{d}(x,y)^2\ d\pi : \pi \text{ transport map from } \nu \text{ to } \sigma\right\},$$

which is the *Wasserstein metric*. The space $(\mathscr{P}(\mathcal{X}), W_2)$ is the *Wasserstein space* and it is a geodesic space. Let $\nu \in \mathscr{P}(\mathcal{X})$ as before. The relative entropy of ν with respect to μ is then given by

$$\operatorname{Ent}_\mu(\nu) = \begin{cases} \int_{\mathcal{X}} \rho \log \rho \, d\mu, & \nu \ll \mu, \quad d\nu = \rho \, d\mu, \\ +\infty, & \text{otherwise.} \end{cases}$$

Suppose that $\nu_0,\ \nu_1 \in \mathscr{P}(\mathcal{X})$ and let ν_t be the geodesic between ν_0 and ν_1. Now, suppose that there exists $\kappa \in \mathbb{R}$ such that

$$\operatorname{Ent}_\mu(\nu_t) \leq (1-t)\operatorname{Ent}_\mu(\nu_0) + t\operatorname{Ent}_\mu(\nu_1) - \frac{\kappa}{2}(1-t)tW_2^2(\nu_0,\nu_1).$$

Then, we say that $(\mathcal{X}, \mathrm{d}, \mu)$ has Ricci curvature bounded below by κ, or is said to be $\mathrm{CD}(\kappa, \infty)$.

For a Lipschitz function $\xi \in \operatorname{Lip}(\mathcal{X}, \mathrm{d})$, recall the *pointwise Lipschitz constant*:

$$\mathbf{Lip}\,\xi(x) = \limsup_{y \to x} \frac{|\xi(x) - \xi(y)|}{\mathrm{d}(x,y)},$$

for non-isolated points $x \in \mathcal{X}$. For $f \in \mathrm{L}^2(\mathcal{X}, \mu)$, if $f_n \to f$ with $f_n \in \operatorname{Lip}(\mathcal{X}, \mathrm{d})$, define the *Cheeger energy*:

$$\mathrm{Ch}(f) = \inf_{\operatorname{Lip}(\mathcal{X},\mathrm{d}) \ni f_n \to f} \lim_{n\to\infty} \frac{1}{2} \int_{\mathcal{X}} |\,\mathbf{Lip}\, f_n|^2 \, d\mu.$$

If no such such sequence exists, $\mathrm{Ch}(f) = +\infty$. The first-order Sobolev space is defined as:

$$\mathrm{W}^{1,2}(\mathcal{X}) = \left\{ f \in \mathrm{L}^2(\mathcal{X}, \mu) : \mathrm{Ch}(f) < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\mathrm{W}^{1,2}}^2 = \|f\|_2^2 + 2\mathrm{Ch}(f).$$

If this norm polarises, i.e., $(\mathrm{W}^{1,2}(\mathcal{X}), \|\cdot\|_{\mathrm{W}^{1,2}})$ is a Hilbert space, then we say that $(\mathcal{X}, \mathrm{d}, \mu)$ is *infinitesimally Hilbertian*. The space $(\mathcal{X}, \mathrm{d}, \mu)$ is RCD if it is $\mathrm{CD}(\kappa, \infty)$ and it is infinitesimally Hilbertian. This is equivalent to the Laplacian associated to the energy Ch being linear.

For an RCD space $(\mathcal{X}, \mathrm{d}, \mu)$, the heat kernel ρ_t exists and it is *Lipschitz*. There is an induced heat action on $(\mathscr{P}(\mathcal{X}), W_2)$, which is a map $\mathrm{H}_t : \mathscr{P}(\operatorname{spt}\,\mu) \to \mathscr{P}(\operatorname{spt}\,\mu)$ such that for all $\nu, \sigma \in \mathscr{P}(\mathcal{X})$ with $\operatorname{spt}\,\nu,\ \operatorname{spt}\,\sigma \subset \operatorname{spt}\,\mu$,

$$W_2(\mathrm{H}_t(\nu), \mathrm{H}_t(\sigma)) \leq \mathrm{e}^{-\kappa t}\, W_2(\nu, \sigma).$$

For $(\mathcal{X}, \mathrm{d}, \mu) = (\mathcal{M}, \mathrm{d_g}, \mu_\mathrm{g}/\mu_\mathrm{g}(\mathcal{M}))$, if $s \mapsto \gamma_s$ is an absolutely continuous curve, then

$$\mathrm{H}_t(\delta_{\gamma_s}) = \rho_t^{\mathrm{g}}(\gamma_s, \cdot\,)\, d\mu_\mathrm{g}.$$

2.2. **The GM flow for RCD spaces.** Define: $\tilde{\mathrm{d}}_t(x,y) = W_2(\mathrm{H}_t(\delta_x), \mathrm{H}_t(\delta_y))$. The spaces $(\mathcal{X}, \tilde{\mathrm{d}}_t)$ are pseudo-metric spaces for each $t > 0$. Noting that $s \to \gamma_s$ is d-Lipschitz implies that it is also $\tilde{\mathrm{d}}_t$ Lipschitz, define

$$\mathrm{d}_t(x,y) = \inf_{\gamma \ \mathrm{d-Lipschitz}} \int |\dot{\gamma_s}|_{\tilde{\mathrm{d}}_t} \, ds,$$

where

$$|\dot{\gamma_s}|_{\tilde{\mathrm{d}}_t} = \lim_{h \to 0} \frac{\tilde{\mathrm{d}}_t(\gamma_{s+h}, \gamma_s)}{h}.$$

The family of spaces $(\mathcal{X}, \mathrm{d}_t)$ are metric spaces for all $t > 0$, $\lim_{t\to 0} \mathrm{d}_t = \mathrm{d}$.

Theorem 2.1 (Gigli-Mantegazza, [GM]). *When* $(\mathcal{X}, \mathrm{d}, \mu) = (\mathcal{M}, \mathrm{d}_{\mathrm{g}}, \mu_{\mathrm{g}}/\mu_{\mathrm{g}}(\mathcal{M}))$, *we have that* $\mathrm{d}_t = \mathrm{d}_{\mathrm{g}_t}$.

Theorem 2.2 (Theorem 1.1, [BLM]). *Let* $\mathcal{M}$ *be a smooth, compact manifold with* rough metric g *that induces a distance metric* d_{g}. *Moreover, suppose there exists* $K \in \mathbb{R}$ *and* $N > 0$ *such that* $(\mathcal{M}, \mathrm{d}_{\mathrm{g}}, \mu_{\mathrm{g}}) \in \mathrm{RCD}(K,N)$. *If* $\mathcal{S} \neq \mathcal{M}$ *is a closed set and* $\mathrm{g} \in \mathrm{C}^k(\mathcal{M} \setminus \mathcal{S})$, *there exists a family of metrics* $\mathrm{g}_t \in \mathrm{C}^{k-1,1}$ *on* $\mathcal{M} \setminus \mathcal{S}$ *evolving according to* (GM) *on* $\mathcal{M} \setminus \mathcal{S}$. *For two points* $x, y \in \mathcal{M}$ *that are* g_t*-admissible, the distance* $\mathrm{d}_t(x,y)$ *given by the* $\mathrm{RCD}(K,N)$ *Gigli-Mantegazza flow is induced by* g_t.

Note: $x, y \in \mathcal{M} \setminus \mathcal{S}$ are g_t-admissible if for any abs. cts. $\gamma : I \to \mathcal{M}$ connecting these points, there is another abs. cts. $\gamma' : I \to \mathcal{M}$ with d_t-length less than γ and for which $\gamma'(s) \in \mathcal{M} \setminus \mathcal{S}$. Let $\mathrm{g} \in \mathbf{\Gamma}(\mathcal{T}^{(2,0)}\mathcal{M})$ be symmetric, with measurable coefficients. Suppose for each $x \in \mathcal{M}$, there exists some chart (ψ, U) containing x and a constant $C \geq 1$ (dependent on U), such that, for y-a.e. in U,

$$C^{-1}|u|_{\psi^*\delta(y)} \leq |u|_{\mathrm{g}(y)} \leq C|u|_{\psi^*\delta(y)},$$

where $u \in \mathrm{T}_y\mathcal{M}$, $|u|^2_{\mathrm{g}(y)} = \mathrm{g}(u,u)$ and $\psi^*\delta$ is the pullback of the Euclidean metric inside $\psi(U) \subset \mathbb{R}^n$. Then, g is called a *rough metric*.

- By the usual expression $d\mu_{\mathrm{g}} = \sqrt{\det \mathrm{g}_{ij}}\, d\mathscr{L}$ inside local comparable charts, obtain a Borel measure μ_{g}, finite on compact sets.
- A priori, there may not be an induced length structure.
- The L^p spaces exist, and differentiation on functions $\nabla = \mathrm{d}$ is densely-defined and closable.
- Sobolev space $\mathrm{W}^{1,2}(\mathcal{M}) = \mathcal{D}(\overline{\nabla})$ and the Laplacian is a self-adjoint operator $\Delta_{\mathrm{g}} = -\operatorname{div}\nabla$, where $\operatorname{div} = \nabla^*$.
- Two rough metrics g and $\tilde{\mathrm{g}}$ are C-close for some $C \geq 1$ if

  $$C^{-1}|u|_{\tilde{\mathrm{g}}(y)} \leq |u|_{\mathrm{g}(y)} \leq C|u|_{\tilde{\mathrm{g}}(y)},$$

  for y-a.e. in $\mathcal{M}$.
- In this situation, $\Delta_{\mathrm{g}} = -\theta^{-1}\operatorname{div}_{\tilde{\mathrm{g}}} \theta \mathrm{B}\nabla$, where

  $$\mathrm{g}(u,v) = \tilde{\mathrm{g}}(\mathrm{B}u, v) \quad \text{and} \quad \theta = \sqrt{\det \mathrm{B}}.$$

Main fact: for $\mathcal{M}$ compact, for every rough metric g, there exists a smooth metric $\tilde{g}$ that is C-close to g. We have that $\varphi_{t,x,v} \in W^{1,2}(\mathcal{M})$ solves:

$$-\operatorname{div}_{g}(\rho^{g}_t(x,y)\nabla\varphi_{t,x,v})(y) = (d_x\rho^{g}_t(x,y))(v)$$
$$\int_{\mathcal{M}} \varphi_{t,x,v}(y)\ d\mu_{g}(y) = 0.$$

if and only if

$$-\operatorname{div}_{\tilde{g}}(B(y)\theta(y)\rho^{g}_t(x,y)\nabla\varphi_{t,x,v})(y) = \theta(y)(d_x\rho^{g}_t(x,y))(v)$$
$$\int_{\mathcal{M}} \varphi_{t,x,v}(y)\ d\mu_{g}(y) = 0.$$

So, it suffices to study divergence form operators with L^∞ coefficients for smooth metrics $\tilde{g}$. Fix $\mathcal{M}$ smooth compact manifold and $\tilde{g}$ a smooth Riemannian metric. Let $A \in \boldsymbol{\Gamma}(L^\infty(\mathcal{T}^{(1,1)}\mathcal{M}))$ real-symmetric and elliptic:

(i) there exist $\kappa > 0$ such that for x a.e. $\tilde{g}_x(A(x)u,u) \geq \kappa|u|^2_x$
(ii) there exists a $\Lambda < \infty$ such that $\operatorname{esssup}_{x\in\mathcal{M}} |A(x)| < \Lambda$.

- Associated energy: $J_A[u,v] = \langle A\nabla u, \nabla v\rangle$ for $\mathcal{D}(J_A) = W^{1,2}(\mathcal{M})$.
- Ellipticity gives: $\kappa\|\nabla u\|^2 \leq J_A[u,u] \leq \Lambda\|\nabla u\|^2$.
- Lax-Milgram theorem yields $L_A = -\operatorname{div} A\nabla$ with domain
$$\mathcal{D}(L_A) = \left\{u \in W^{1,2}(\mathcal{M}) : v \mapsto J_A[u,v]\text{continuous}\right\}$$
as a non-negative self-adjoint operator. Moreover, $\mathcal{D}(\sqrt{L_A}) = W^{1,2}(\mathcal{M})$.
- $L^2(\mathcal{M}) = \mathcal{N}(L_A) \oplus^\perp \overline{\mathcal{R}(L_A)}$,
- $\mathcal{N}(L_A) = \mathcal{N}(\nabla)$ and crucially,
$$\overline{\mathcal{R}(L_A)} = \mathcal{R} := \left\{u \in L^2(\mathcal{M}) : \int u = 0\right\},$$
- The operator $L_A^{\mathcal{R}} = L_A$ with $\mathcal{D}(L_A^{\mathcal{R}}) = \mathcal{D}(L_A) \cap \mathcal{R}$ is an unbounded operator $L_A^{\mathcal{R}} : \mathcal{R} \to \mathcal{R}$.
- $\sigma(L_A) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\}$, and
- $\sigma(L_A^{\mathcal{R}}) = \{0 < \lambda_1 \leq \lambda_2 \leq \dots\}$.

Proposition 2.3. Let $f \in L^2(\mathcal{M})$ with $\int f\ d\mu_{\tilde{g}} = 0$. Then, there exists a unique $u \in W^{1,2}(\mathcal{M})$ with $\int u\ d\mu_{\tilde{g}} = 0$ such that $L_A u = f$. Explicitly, $u = (L_A^{\mathcal{R}})^{-1} f$.

Let g be a rough metric, $g(u,v) = \tilde{g}(Bu,v)$, and let $(x,y) \mapsto \omega_x(y) \in C^{0,1}(\mathcal{M}^2)$, $\omega_x > 0$. Suppose there exists $\varnothing \neq \mathcal{N} \subset \mathcal{M}$ open set on which $x \mapsto \omega_x(\cdot) \in C^k(\mathcal{N})$, for $k \geq 1$. Let

$$D_x = -\operatorname{div}_{g} \omega_x\nabla = -\theta^{-1}\operatorname{div}_{\tilde{g}} B\theta\omega_x\nabla.$$

The continuity equation is then

$$D_x\varphi_x = \eta_x. \tag{F}$$

Proposition 2.4. Let $\eta_x \in L^2(\mathcal{M})$ with $\int \eta_x\ d\mu_{g} = 0$. Then there exists a unique $\varphi_x \in W^{1,2}(\mathcal{M})$ with $\int \varphi_x\ d\mu_{g} = 0$ solving (F).

To understand regularity, we need to understand the behaviour of the operators $x \mapsto \mathrm{D}_x$. Two crucial facts:

- $\mathcal{D}(\mathrm{D}_x) = \mathcal{D}(\Delta_{\mathrm{g}})$ and $\mathrm{D}_x u = \omega_x \Delta_{\mathrm{g}} u - \mathrm{g}(\nabla u, \nabla \omega_x)$,
- $\mathcal{M} \ni x \mapsto \mathrm{D}_x : (\mathcal{D}(\Delta_{\mathrm{g}}), \|\cdot\|_{\Delta_{\mathrm{g}}}) \to \mathrm{L}^2(\mathcal{M})$ is a uniformly bounded family of operators and $\|u\|_{\mathrm{D}_x} \simeq \|u\|_{\Delta_{\mathrm{g}}}$ holds with the implicit constant independent of $x \in \mathcal{M}$.

Let $v \in \mathrm{T}_x\mathcal{M}$ and $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\gamma(0) = x$ and $\dot\gamma(0) = v$. Let $f : \mathcal{N} \to \mathcal{V}$, where $\mathcal{V}$ where $\mathcal{V}$ is some normed vector space.

- Difference quotient: $Q^v_s f(x) = \frac{f(x) - f(\gamma(s))}{s}$.
- *Directional derivative* of f (when it exists and it is independent of the generating curve γ): $(\mathrm{d}_x f(x))(v) = \lim_{s\to 0} Q^v_s f(x)$.

For us, $\mathcal{V} = \mathrm{L}^2(\mathcal{M})$ with the weak topology for the choice $f(x) = \mathrm{D}_x$. More precisely, if there exists $\tilde{\mathrm{D}}_x : \mathcal{D}(\Delta_{\mathrm{g}}) \to \mathrm{L}^2(\mathcal{M})$ satisfying $\lim_{s\to 0}\langle Q^v_s \mathrm{D}_x u, w\rangle = \langle \tilde{\mathrm{D}}_x u, w\rangle$, for every $w \in \mathrm{W}^{1,2}(\mathcal{M})$, say that D_x has a *(weak) derivative* at x and write $(\mathrm{d}_x \mathrm{D}_x) = \tilde{\mathrm{D}}_x$.

Proposition 2.5. Let $x \mapsto u_x : \mathcal{N} \to \mathcal{D}(\Delta_{\mathrm{g}})$, $v \in \mathrm{T}_x\mathcal{M}$ and suppose that $(\mathrm{d}_x u_x)(v)$ exists weakly. Then $(\mathrm{d}_x \mathrm{D}_x u_x)(v)$ exists weakly if and only if $\mathrm{D}_x((\mathrm{d}_x u_x)(v))$ exists weakly and

$$(\mathrm{d}_x \mathrm{D}_x u_x)(v) = (\mathrm{d}_x \mathrm{D}_x)(v) u_x + \mathrm{D}_x((\mathrm{d}_x u_x)(v)).$$

Hence,

Theorem 2.6. *Suppose that $k \geq 1$ and $(x, y) \mapsto \omega_x(y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$ and $x \mapsto \omega_x \in \mathrm{C}^k(\mathcal{N})$. Moreover, suppose that $(x, y) \mapsto \eta_x(y) \in \mathrm{C}^0(\mathcal{N} \times \mathcal{M})$ and $x \mapsto \eta_x(y) \in \mathrm{C}^l(\mathcal{N})$ where $l \geq 1$. If at $x \in \mathcal{N}$, φ_x solves* (F) *with $\int_{\mathcal{M}} \varphi_x \, d\mu_{\mathrm{g}} = \int_{\mathcal{M}} \eta_x \, d\mu_{\mathrm{g}} = 0$, the map $x \mapsto \langle \eta_x, \varphi_x\rangle \in \mathrm{C}^{\min\{k,l\}-1,1}(\mathcal{N})$.*

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