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Errata to “Smooth convergence away from singular sets”

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Seven years after the publication of “Smooth convergence away from singular sets” [LS13], Brian Allen discovered a counter example to the published statement of Theorem 1.3. Note that Theorem 4.6 (which is the key theorem cited in other papers) remains correct. We have added an hypothesis to correct the statement of Theorem 1.3 and its consequences, and provide a detailed proof and explanation of the error within as well as presenting Brian Allen’s example in the Appendix. We have also made corrections to the arxiv posting of this paper.

1. Introduction

We regret to report that seven years after the publication of “Smooth convergence away from singular sets” [LS13], Brian Allen discovered a counter example to the published statement of Theorem 1.3. We present his counter example in the appendix to this errata. Theorem 1.3 is false as stated in the original publication for smooth convergence $g_j \to g_\infty$ on $M \setminus S$ where the convergence is only uniform on compact sets $K \subset M \setminus S$. We are able to correct this theorem and its consequences, Theorem 1.2 and Theorem 6.6, by adding in the assumption that the convergence of $g_j \to g_\infty$ is also uniform from below on $M \setminus S$ in the sense described in the following revision of Definition 1.1:

Definition 1.1. We will say that a sequence of Riemannian metrics, g_j, on a compact manifold, M, converges smoothly away from $S \subset M$ to a Riemannian metric g_∞ on $M \setminus S$ if for every compact set $K \subset M \setminus S$, g_j converge $C^{k,\alpha}$ smoothly to g_∞ as tensors. In addition we say that it converges uniformly from below if there exists $\delta_j \to 0$ such that $g_j \geq (1-\delta_j)^2 g_\infty$ on $M \setminus S$.

Using this new hypothesis we can prove Theorem 1.2 and Theorem 1.3 stated as follows:

Theorem 1.2. *Let $M_i = (M, g_i)$ be a sequence of oriented compact Riemannian manifolds with uniform lower Ricci curvature bounds,*

$$\mathrm{Ricci}_{\mathrm{g_i}}(\mathrm{V},\mathrm{V}) \geq (\mathrm{n}-1)\mathrm{H}\,\mathrm{g_i}(\mathrm{V},\mathrm{V}) \qquad \forall \mathrm{V} \in \mathrm{TM_i} \tag{1.1}$$

which converges smoothly away from S uniformly from below where S is a submanifold of codimension 2.

If there is a connected precompact exhaustion, W_j, of $M \setminus S$,

$$\bar{W}_j \subset W_{j+1} \text{ with } \bigcup_{j=1}^{\infty} W_j = M \setminus S \tag{1.2}$$

satisfying

$$\mathrm{diam}(M_i) \leq D_0, \tag{1.3}$$

$$\mathrm{Vol}_{g_i}(\partial W_j) \leq A_0, \tag{1.4}$$

and

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j\to\infty} V_j = 0, \tag{1.5}$$

then

$$\lim_{j\to\infty} d_{GH}(M_j, N) = 0, \tag{1.6}$$

where N is the metric completion of $(M \setminus S, g_\infty)$.

Theorem 1.3. *Let $M_i = (M, g_i)$ be a sequence of compact oriented Riemannian manifolds such that there is a submanifold, S, of codimension 2, and connected precompact exhaustion, W_j, of $M \setminus S$ satisfying (1.2) with g_i converge smoothly to g_∞ on $M \setminus S$ uniformly from below such that*

$$\mathrm{diam}_{M_i}(W_j) \leq D_0 \qquad \forall i \geq j, \tag{1.7}$$

$$\mathrm{Vol}_{g_i}(\partial W_j) \leq A_0, \tag{1.8}$$

and

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \leq V_j \text{ where } \lim_{j\to\infty} V_j = 0. \tag{1.9}$$

Then

$$\lim_{j\to\infty} d_{\mathcal{F}}(M'_j, N') = 0. \tag{1.10}$$

where N' is the settled completion of $(M \setminus S, g_\infty)$.

Note that Brian Allen's counter example is a counter example to Theorem 1.3. We conjecture that Theorem 1.2 is true as originally stated, but we leave that to future mathematicians to study.

The error in the proof of Theorem 1.3 was traced by Brian Allen to a reversal of indices in limits in the original proof of Theorem 5.2. We find that by correcting the order of the limits in Definition 5.1 of uniform well embeddedness, we can prove Theorem 5.2 as originally stated. This is reviewed in detail within.

We also correct the proof of Lemma 5.7 to adapt to this new definition of uniform well embeddedness using the notion of smooth convergence away from a singular set uniformly from below. Thus Theorem 1.3 and its consequences (Theorem 1.2 and Theorem 6.6) are true assuming this stronger hypothesis. This is reviewed within as well.

This paper has been cited many times since its publication. We believe the only paper that needs revision is [L16] by the first author of this paper. The other papers apply only Theorem 4.6, which remains correct as originally stated and proven. To make this errata as easy to read as possible, we break it into the same sections and subsections as the original paper. We have also posted a version 4 of this article on the arxiv where all these corrections have been made in blue exactly where they belong in the original 63 page article. We apologize for the necessity and for the length of this errata.

2. Background

This section is correct as written in the original paper.

3. Examples

The examples in this section were rechecked carefully and are all correct as presented in the original paper.

4. Explicit estimates with isometric embeddings

The work in this section is correct as originally stated and proven including the essential Theorem 4.6 that has been applied in a number of papers.

5. Intrinsic flat limits

The limits in the following restatement of Definition 5.1 have been reordered to match what we need to prove Theorem 5.2 with its original proof.

Definition 5.1. Given a sequence of Riemannian manifolds $M_i = (M, g_i)$ and an open subset, $U \subset M$, a connected precompact exhaustion, W_j, of U satisfying (1.2) is *uniformly well embedded* if

$$\lambda_{i,j,k} = \sup_{x,y \in W_j} |d_{(W_k, g_i)}(x, y) - d_{(M, g_i)}(x, y)| \tag{5.1}$$

has

$$\limsup_{j\to\infty} \limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} = 0. \tag{5.2}$$

and thus a uniform upper bound

$$\lambda_{i,j,k} \le \lambda_0 < \infty \tag{5.3}$$

This theorem is correct as originally stated using this new definition. However the proof which appeared after the statement of Lemma 5.6 has significant changes which we will describe below after reviewing the material leading up to it so we include its statement here so that it is easy to follow the new proof.

Theorem 5.2. *Let $M_i = (M, g_i)$ be a sequence of compact oriented Riemannian manifolds such that there is a closed subset, S, and a uniformly well embedded connected precompact exhaustion, W_j, of $M \setminus S$ satisfying (1.2) such that g_i converge smoothly to g_∞ on each W_j with*

$$\mathrm{diam}_{M_i}(W_j) \le D_0 \qquad \forall i \ge j, \tag{5.4}$$

$$\mathrm{Vol}_{g_i}(\partial W_j) \le A_0 \tag{5.5}$$

and

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \le V_j \text{ where } \lim_{j\to\infty} V_j = 0. \tag{5.6}$$

Then

$$\lim_{j\to\infty} d_{\mathcal{F}}(M'_j, N') = 0 \tag{5.7}$$

where N' is the settled completion of $N = (M \setminus S, g_\infty)$.

Remark 5.3. This remark about the examples is correct as originally stated and now we also have the example by Brian Allen in the Appendix to justify why we changed the definition of uniform well embeddedness.

5.1. Creating spaces from exhaustions: has minor corrections

Proposition 5.4. *This proposition is correct as originally stated and the proof has minor typos at the end of the proof which can be corrected as follows:*

$$\begin{aligned} d_{W_k}(x_{i,k}, y_{i,k}) &\le d_{W_k}(x_{i_k}, x_i) + d_{W_k}(x_i, y_i) + d_{W_k}(y_i, y_{i_k}) \\ &< d_N(x_i, y_i) + 3\epsilon'/5 \\ &\le d_N(x_{i_k}, x_i) + d_N(x_i, y_i) + d_N(y_i, y_{i_k}) + 3\epsilon'/5 \\ &\le d_{W_k}(x_{i_k}, x_i) + d_N(x_i, y_i) + d_{W_k}(y_i, y_{i_k}) + 3\epsilon'/5 \\ &\le d_{W_k}(x_{i_k}, x_i) + d_N(x_i, y_i) + d_{W_k}(y_i, y_{i_k}) + 3\epsilon'/5 \\ &< d_N(x_i, y_i) + 5\epsilon'/5 = d_N(x_i, y_i) + \epsilon'. \end{aligned}$$

Since $d_N(x_{i,k}, y_{i,k}) \le d_{W_k}(x_{i,k}, y_{i,k})$, we have

$$\lambda_{i,k} < d_{W_k}(x_{i,k}, y_{i,k}) - d_N(x_{i,k}, y_{i,k}) < \epsilon'. \tag{5.8}$$

Example 5.5. This example is correct as originally presented.

5.2. Proof of Theorem 5.2 has essential corrections

Lemma 5.6. *This lemma is correct as originally stated and proven.*

We now present the corrected proof of Theorem 5.2 starting as in the original paper, pointing out the error, and continuing with the correction:

Proof. By hypothesis (5.6) and Lemma 5.6 we have:

$$\operatorname{Vol}(M_i) \le V_0, \tag{5.9}$$

Next we prove that (W_j, g_∞) satisfy the hypothesis of Proposition 5.4. Observe that hypothesis (5.6) and smooth convergence we have

$$\operatorname{Vol}_{g_\infty}(W_j) = \lim_{i\to\infty} \operatorname{Vol}_{g_i}(W_j) \le V_0, \tag{5.10}$$

while (5.5) implies

$$\operatorname{Vol}_{g_\infty}(\partial W_j) = \lim_{i\to\infty} \operatorname{Vol}_{g_i}(\partial W_j) \le A_0. \tag{5.11}$$

Finally

$$\begin{aligned}
\operatorname{diam}_N(N) &= \lim_{j\to\infty} \operatorname{diam}_N(W_j) && (5.12)\\
&\le \lim_{j\to\infty} \lim_{k\to\infty} \operatorname{diam}_{(W_k,g_\infty)}(W_j) && (5.13)\\
&\le \lim_{j\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \operatorname{diam}_{(W_k,g_i)}(W_j) && (5.14)\\
&\le \lim_{j\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} \operatorname{diam}_{(M,g_i)}(W_j) + \lambda_{i,j,k} && (5.15)\\
&\le \limsup_{j\to\infty} \limsup_{k\to\infty} \limsup_{i\to\infty} D_0 + \lambda_{i,j,k} && (5.16)\\
&\le D_0 + \lambda_0. && (5.17)
\end{aligned}$$

Thus by Proposition 5.4 we have

$$d_{\mathcal{F}}\big((W_j, g_\infty), (N', d_\infty)\big) = F_j \text{ where } \lim_{j\to\infty} F_j = 0. \tag{5.18}$$

Next we will apply Theorem 4.6 to show $M_1 = (W_k, g_\infty)$ and $M_2 = (M, g_i)$ are close in the intrinsic flat sense by setting $U_1 = W_j \subset W_k$ and $U_2 = W_j \subset M$ for some well chosen $j < k$ Then the values in the hypothesis of the theorem are

$$\epsilon = \epsilon_{i,j} \text{ where } \lim_{i\to\infty} \epsilon_{i,j} = 0, \tag{5.19}$$

$$D_{U_2} \le \operatorname{diam}_{(M,g_i)}(W_j) \le D_0 \tag{5.20}$$

$$D_{U_1} \le \operatorname{diam}_{(W_k,g_i)}(W_j) \le D_0 + \lambda_0 \tag{5.21}$$

$$a = a_{i,j} \le a_{i,j} = 2(D_0+\lambda_0)\arccos(1+\epsilon_{i,j})^{-1}/\pi \tag{5.22}$$

$$\lambda = \lambda'_{i,j,k} \text{ instead of } \lambda_{i,j,k} \tag{5.23}$$

$$h = h_{i,j,k} \le \sqrt{\lambda'_{i,j,k}(D_0+\lambda_0+\lambda'_{i,j,k}/4)} \tag{5.24}$$

$$\bar{h} = \bar{h}_{i,j,k} \le \max\{h_{i,j,k}, \sqrt{\epsilon_{i,j}^2 + 2\epsilon_{i,j}(D_0+\lambda_0)}\} \tag{5.25}$$

Thus

$$d_{\mathcal{F}}\big((W_k,g_\infty),(M,g_i)\big) \le \big(\bar{h}_{i,j,k} + a_{i,j}\big)\Big(2V_0 + 2A_0\Big) + 2V_j. \tag{5.26}$$

Brian Allen observed the above estimate was incorrect in the published version because in (5.23) we had

$$\lambda_{i,j,k} = \sup_{x,y\in W_j} |d_{(W_k,g_i)}(x,y) - d_{(M,g_i)}(x,y)| \text{ as in (5.1)}$$

but to apply Theorem 4.6 we need

$$\lambda'_{i,j,k} = \sup_{x,y\in W_j} |d_{(W_k,g_\infty)}(x,y) - d_{(M,g_i)}(x,y)|.$$

We observe now that

$$|\lambda'_{i,j,k} - \lambda_{i,j,k}| \le \eta_{i,j,k}$$

where

$$\eta_{i,j,k} = \sup_{x,y\in W_j} |d_{(W_k,g_i)}(x,y) - d_{(W_k,g_\infty)}(x,y)|.$$

So by the smooth convergence of g_i to g_∞ on W_k we have

$$(1+\epsilon_{i,k})^{-2} g_\infty \le g_i \le (1+\epsilon_{i,k})^2 g_\infty \text{ on } W_k \text{ where } \lim_{i\to\infty} \epsilon_{i,k} = 0$$

Thus for any curve, C, in W_k we have

$$(1+\epsilon_{i,k})^{-1} L_{g_\infty}(C) \le L_{g_i}(C) \le (1+\epsilon_{i,k}) L_{g_\infty}(C)$$

Applying this to a g_i-minimizing curve C_i from x to y in W_k we have

$$\begin{aligned} d_{(W_k,g_\infty)}(x,y) &\le L_{g_\infty}(C_i) \le (1+\epsilon_{i,k})L_{g_i}(C_i) \\ &= (1+\epsilon_{i,k})d_{(W_k,g_i)}(x,y) \\ &\le d_{(W_k,g_i)}(x,y) + \epsilon_{i,k}(D_0+\lambda_0) \end{aligned}$$

and applying this to a g_∞-minimizing curve C_∞ from x to y in W_k we have

$$\begin{aligned} d_{(W_k,g_i)}(x,y) &\le L_{g_i}(C_\infty) \le (1+\epsilon_{i,k})L_{g_\infty}(C_\infty) \\ &= (1+\epsilon_{i,k})d_{(W_k,g_\infty)}(x,y) \\ &\le d_{(W_k,g_\infty)}(x,y) + \epsilon_{i,k}(1+\epsilon_{i,k})(D_0+\lambda_0) \end{aligned}$$

because

$$d_{(W_k,g_\infty)}(x,y) \le (1+\epsilon_{i,k})d_{(W_k,g_i)}(x,y) \le (1+\epsilon_{i,k})(D_0+\lambda_0)$$

Thus

$$\eta_{i,j,k} \le \eta_{i,k} = \epsilon_{i,k}(1+\epsilon_{i,k})(D_0+\lambda_0)(D_0+\lambda_0)$$

and for fixed k,

$$\lim_{i\to\infty} \eta_{i,k} = 0.$$

So

$$\lim_{i\to\infty} \lambda'_{i,j,k} = \lim_{i\to\infty} \lambda_{i,j,k}.$$

This leads to the reordering of the limits in our fixed definition of uniformly well embedded:

$$\limsup_{j\to\infty}\limsup_{k\to\infty}\limsup_{i\to\infty} \lambda_{i,j,k} = 0$$

which will imply

$$\limsup_{j\to\infty}\limsup_{k\to\infty}\limsup_{i\to\infty} \lambda'_{i,j,k} = 0$$

and thus

$$\limsup_{j\to\infty}\limsup_{k\to\infty}\limsup_{i\to\infty} \bar{h}_{i,j,k} = 0.$$

Combining (5.26) with (5.18) we have for any $j<k$,

$$d_{\mathcal{F}}\big((N,g_\infty),(M,g_i)\big) \le \big(\bar{h}_{i,j,k}+a_{i,j}\big)\Big(2V_0+2A_0\Big) + 2V_j + F_j. \tag{5.27}$$

So now we should take $i \to \infty$ first. Recall that for any fixed j, $\lim_{i\to\infty} \epsilon_{i,j} = 0$, thus $\lim_{i\to\infty} a_{i,j} = 0$ as well.

$$\limsup_{i\to\infty} d_{\mathcal{F}}\big((N', g_\infty), (M, g_i)\big) \leq \big(\bar{h}_{j,k} + 0\big)\left(2V_0 + 2A_0\right) + 2V_j + F_j + 0.$$

where $\bar{h}_{j,k} = \limsup_{i\to\infty} \bar{h}_{i,j,k}$. Next taking the limsup as $k \to \infty$

$$\limsup_{i\to\infty} d_{\mathcal{F}}\big((N', g_\infty), (M, g_i)\big) \leq \big(\bar{h}_{j} + 0\big)\left(2V_0 + 2A_0\right) + 0 + 0.$$

where $\bar{h}_j = \limsup_{k\to\infty} \bar{h}_{j,k}$. Taking the limsup as $j \to \infty$

$$\limsup_{i\to\infty} d_{\mathcal{F}}\big((N', g_\infty), (M, g_i)\big) \leq (0 + 0)\left(2V_0 + 2A_0\right) + 0 + 0 = 0.$$

□

5.3. Codimension 2 singular sets has essential corrections

The following lemma combined with Theorem 5.2 is needed to complete the proof of Theorem 1.3. Note that it has both a new statement and a new proof:

Lemma 5.7. *Let M be compact, $g_i \to g_\infty$ smoothly away from S uniformly from below where S is a closed submanifold of codimension 2, $\mathrm{diam}_{g_\infty}(M \setminus S) < \infty$, and $\mathrm{diam}_{g_i}(M) \leq D_0$ then, any connected precompact exhaustion, W_j, of $M \setminus S$ is uniformly well embedded.*

With the correction to Definition 5.1 the original proof of this lemma is no longer correct. We now prove this lemma using the new definition of smooth convergence away from S uniformly from below and the adapted definition of uniformly well embedded. The proof is similar to the original proof but we must be careful to take the limits in the correct order.

Proof. Observe that

$$d_{(W_k,g_i)}(x,y) - d_{(M,g_i)}(x,y) \geq 0 \tag{5.28}$$

because $W_k \subset M$ and so

$$d_{(W_k,g_i)}(x,y) = \inf\{L_{g_i}(C) \mid C : [0,1] \to W_k,\ C(0) = x,\ C(1) = y\} \tag{5.29}$$

$$\geq \inf\{L_{g_i}(C) \mid C : [0,1] \to M,\ C(0) = x,\ C(1) = y\} \tag{5.30}$$

$$= d_{(M,g_i)}(x,y). \tag{5.31}$$

Thus

$$\lambda_{i,j,k} = \sup_{x,y\in W_j} d_{(W_k,g_i)}(x,y) - d_{(M,g_i)}(x,y). \tag{5.32}$$

Since $\bar{W}_j$ is compact, there exists $x_{i,j,k}, y_{i,j,k} \in \bar{W}_j$ achieving this supremum:

$$\lambda_{i,j,k} = d_{(W_k,g_i)}(x_{i,j,k}, y_{i,j,k}) - d_{(M,g_i)}(x_{i,j,k}, y_{i,j,k}). \tag{5.33}$$

Consider a subsequence $i' \to \infty$ such that

$$\lim_{i'\to\infty} \lambda_{i',j,k} = \limsup_{i\to\infty} \lambda_{i,j,k} \tag{5.34}$$

and consider a further subsequence, also denoted i' such that

$$x_{i',j,k} \to x_{\infty,j,k} \text{ and } y_{i',j,k} \to y_{\infty,j,k} \in \bar{W}_j. \tag{5.35}$$

In particular, as $i' \to \infty$ for fixed j, k, we have

$$d_{g_\infty,W_k}(x_{i',j,k}, x_{\infty,j,k}) \to 0 \text{ and } d_{g_\infty,W_k}(y_{i',j,k}, y_{\infty,j,k}) \to 0 \tag{5.36}$$

Since $g_i \to g_\infty$ on W_k for fixed k, there exists $H_{i,j,k} > 1$ such that

$$H_{i,j,k}^{-1} \geq \frac{d_{g_i,W_k}(p,q)}{d_{g_\infty,W_k}(p,q)} \geq H_{i,j,k} \qquad \forall p,q \in W_j \tag{5.37}$$

where

$$\lim_{i\to\infty} H_{i,j,k} = 1 \text{ for fixed } j,k. \tag{5.38}$$

Thus as $i' \to \infty$ we have

$$d_{g_i',W_k}(x_{i',j,k}, x_{\infty,j,k}) \leq H_{i',j,k} \cdot d_{g_\infty,W_k}(x_{i',j,k}, x_{\infty,j,k}) \to 1 \cdot 0 = 0 \tag{5.39}$$

and

$$d_{g_i',W_k}(y_{i',j,k}, y_{\infty,j,k}) \leq H_{i',j,k} \cdot d_{g_\infty,W_k}(y_{i',j,k}, y_{\infty,j,k}) \to 1 \cdot 0 = 0 \tag{5.40}$$

Combining these with the triangle inequality we have

$$|d_{(W_k,g_{i'})}(x_{i',j,k}, y_{i',j,k}) - d_{(W_k,g_{i'})}(x_{\infty,j,k}, y_{\infty,j,k})| \to 0. \tag{5.41}$$

Note in addition that

$$d_{g_{i'},M}(p,q) \le d_{g_{i'},W_k}(p,q) \tag{5.42}$$

so as $i' \to \infty$ for fixed j,k we have

$$d_{g_{i'},M}(x_{i',j,k}, x_{\infty,j,k}) \to 0 \text{ and } d_{g_{i'},M}(y_{i',j,k}, y_{\infty,j,k}) \to 0. \tag{5.43}$$

Combining these with the triangle inequality we have

$$|d_{(M,g_{i'})}(x_{i',j,k}, y_{i',j,k}) - d_{(M,g_{i'})}(x_{\infty,j,k}, y_{\infty,j,k})| \to 0. \tag{5.44}$$

Thus

$$\begin{aligned}\limsup_{i\to\infty} \lambda_{i,j,k} = \lim_{i'\to\infty} & d_{(W_k,g_{i'})}(x_{\infty,j,k}, y_{\infty,j,k}) \\ & - d_{(M,g_{i'})}(x_{\infty,j,k}, y_{\infty,j,k}).\end{aligned} \tag{5.45}$$

Let $\gamma_{i',j,k}$ be a $g_{i'}$ minimizing geodesic in M between $x_{\infty,j,k}$ and $y_{\infty,j,k}$. Since S is a submanifold of codimension 2, for any $h_{i'} \in (0, D_0)$, we can find a curve $C_{i',j,k} : [0,1] \to M \setminus S$ between these points such that

$$|L_{g_{i'}}(C_{i',j,k}) - d_{M,g_{i'}}(x_{\infty,j,k}, y_{\infty,j,k})| < h_{i'} \tag{5.46}$$

by sliding $\gamma_{i',j,k}$ over slightly to avoid S. By the new definition of smooth convergence away from S uniformly from below we have

$$g_i \ge (1-\delta_i)^2 g_\infty \text{ on } M \setminus S. \tag{5.47}$$

Thus

$$d_{M,g_{i'}}(x_{\infty,j,k}, y_{\infty,j,k}) \ge (1-\delta_i) L_{g_\infty}(C_{i',j,k}) - h_{i'} \tag{5.48}$$

$$\ge (1-\delta_i) d_{(M\setminus S, g_\infty)}(x_{\infty,j,k}, y_{\infty,j,k}) - h_{i'}. \tag{5.49}$$

Since we can choose $\lim_{i'\to\infty} h_{i'} = 0$ and we have $\delta_i \to 0$,

$$\lim_{i'\to\infty} d_{M,g_{i'}}(x_{\infty,j,k}, y_{\infty,j,k}) \ge d_{(M\setminus S, g_\infty)}(x_{\infty,j,k}, y_{\infty,j,k}). \tag{5.50}$$

Since $g_i \to g_\infty$ uniformly on $\bar{W}_k$, we also have

$$\lim_{i'\to\infty} d_{(W_k,g_{i'})}(x_{\infty,j,k}, y_{\infty,j,k}) = d_{(W_k,g_\infty)}(x_{\infty,j,k}, y_{\infty,j,k}). \tag{5.51}$$

Combining these we have

$$\limsup_{i\to\infty} \lambda_{i,j,k} \le d_{(W_k,g_\infty)}(x_{\infty,j,k}, y_{\infty,j,k}) - d_{(M\setminus S,g_\infty)}(x_{\infty,j,k}, y_{\infty,j,k}). \tag{5.52}$$

Now choose a subsequence k' such that

$$\limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} = \lim_{k'\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \tag{5.53}$$

and choose a further subsequence k' such that

$$x_{\infty,j,k} \to x_{\infty,j} \subset \bar{W}_j \text{ and } y_{\infty,j,k} \to y_{\infty,j} \subset \bar{W}_j \tag{5.54}$$

By the fact that $\bar{W}_j \subset W_k \subset M \setminus S$ and the triangle inequality,

$$\begin{aligned} \limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \le \limsup_{k'\to\infty} d_{(W_{k'},g_\infty)}(x_{\infty,j}, y_{\infty,j}) \\ - d_{(M\setminus S,g_\infty)}(x_{\infty,j}, y_{\infty,j}). \end{aligned} \tag{5.55}$$

For any $\epsilon_j > 0$ we have a curve $C_j : [0,1] \to M \setminus S$ running from $C_j(0) = x_{\infty,j}$ to $C_j(1) = y_{\infty,j}$ such that

$$L_{g_\infty}(C_j) < d_{(M\setminus S,g_\infty)}(x_{\infty,j}, y_{\infty,j}) + \epsilon_j. \tag{5.56}$$

Since $W_{k'}$ exhaust $M \setminus S$, for k' sufficiently large depending on j we have $C_j([0,1]) \subset W_{k'}$, so

$$d_{(W_{k'},g_\infty)}(x_{\infty,j}, y_{\infty,j}) \le L_{g_\infty}(C_j). \tag{5.57}$$

Thus

$$\limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \le \epsilon_j. \tag{5.58}$$

Finally we apply the fact that we can choose $\epsilon_j \to 0$ so that

$$\limsup_{j\to\infty} \limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \le \epsilon_j. \tag{5.59}$$

□

6. Intrinsic flat to GH convergence

The following theorem needs the same stronger hypothesis that Theorems 1.2 and 1.3 needed:

Theorem 6.1. *Let $M_i = (M, g_i)$ be a sequence of oriented compact Riemannian manifolds with a uniform linear contractibility function, ρ, which converges smoothly away from a codimension two submanifold, S, uniformly from below. If there is a connected precompact exhaustion of $M \setminus S$ as in (1.2) satisfying the volume conditions*

$$\mathrm{Vol}_{g_i}(\partial W_j) \le A_0 \tag{6.1}$$

and

$$\mathrm{Vol}_{g_i}(M \setminus W_j) \le V_j \text{ where } \lim_{j\to\infty} V_j = 0, \tag{6.2}$$

then

$$\lim_{j\to\infty} d_{GH}(M_j, N) = 0, \tag{6.3}$$

where N is the settled and metric completion of $(M \setminus S, g_\infty)$.

It's proof follows as before applying the following theorem which is now true using the new definition of uniformly well embedded:

Theorem 6.2. *Let $M_i = (M, g_i)$ be a sequence of compact oriented Riemannian manifolds with a uniform linear contractibility function, ρ, which converges smoothly away from a singular set, S. If there is a uniformly well embedded connected precompact exhaustion of $M \setminus S$ as in (1.2) satisfying the volume conditions (6.1) and (6.2) then*

$$\lim_{j\to\infty} d_{GH}(M_j, N) = 0, \tag{6.4}$$

where N is the settled and metric completion of $(M \setminus S, g_\infty)$.

This theorem's proof follows as before.

The rest of the lemmas and theorems in this section are true as originally stated and proven.

We conjecture that Theorem 6.1 is true as originally stated.

7. Appendix: Example of Brian Allen

Brian Allen sketched out this example to the second author and we have filled in the details. This example is highly technical and understanding the convergence requires modern methods developed by Brian Allen with the second author in [AS19].

Example 7.1. Let g_0 the standard flat metric on $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. Let

$$S = \mathbb{S}^1 \times \{0\} \times \{0\} \subset M \tag{7.1}$$

which is a submanifold of codimension 2. Let $r : M \to [0, \infty)$ be the distance function from S:

$$r(x) = \min\{d_{g_0}(x, y) : y \in S\} \tag{7.2}$$

and let $h_i : [0, \infty) \to [1/2, 1]$ be a smooth nonincreasing function which satisfies

$$h_i(r) = 1/2 \text{ for } r \le 1/i \text{ and } h_i(r) = 1 \text{ for } r \ge 2/i. \tag{7.3}$$

Taking

$$g_i = h_i(r(x))^2 g_0 \tag{7.4}$$

we have a sequence of Riemannian metrics on M such that $g_i \to g_0$ smoothly on compact sets in $M \setminus S$. [1]

We claim that

$$\text{the metric completion of } (M \setminus S, d_{g_0}) \text{ is isometric to } (M, d_{g_0}). \tag{7.5}$$

This can be seen since any geodesics in (M, d_{g_0}) can be approximated by curves in $(M \setminus S, d_{g_0})$ that are arbitrarily close in length since S has codimension 2. Observe however that by the triangle inequality,

$$d_{g_i}(p, q) \le d_{g_i}(p, p') + d_{g_i}(p', q') + d_{g_i}(q', q), \tag{7.6}$$

Since $g_i \le g_0$ everywhere and $g_i = (1/2)^2 g_0$ on S and S is a convex set with respect to g_0, we have

$$d_{g_i}(p, q) \le d_\infty(p, q) \tag{7.7}$$

[1] Since $\sup_{x \in M \setminus S} h_i(r(x)) - 1 = 1/2$ for all $i \in \mathbb{N}$, we see that g_i does not converge to g_0 on $M \setminus S$ uniformly from below.

where

$$d_\infty(p,q) = \min\{d_{g_0}(p,q),\ d_{g_0}(p,p') + (1/2)d_{g_0}(p',q') + d_{g_0}(q',q) : p',q' \in S\}. \tag{7.8}$$

On the other hand we claim

$$d_\infty(p,q) \geq d_{g_i}(p,q) - 3/j. \tag{7.9}$$

To see this take C_i a g_i-minimizing geodesic from p to q, and take p_i the first point on C_i where it enters $r^{-1}[0,1/i]$ and q_i to be the last point in that set. Then since $g_i \geq (1/2)^2 g_0$ on $r^{-1}[0,1/j]$ and $g_i = g_0$ elsewhere we have

$$d_{g_i}(p,q) = d_{g_i}(p,p_j) + d_{g_i}(p_i,q_i) + d_{g_i}(q_i,q) \tag{7.10}$$
$$\geq d_{g_0}(p,p_i) + (1/2)d_{g_0}(p_i,q_i) + d_{g_0}(q_i,q) \tag{7.11}$$

Taking $p_i', q_j' \in S$ closest to p_i, q_i respectively, we know

$$d_{g_0}(p_i',p_i) \leq 1/i \text{ and } d_{g_0}(q_i',q_i) \leq 1/i. \tag{7.12}$$

So

$$d_{g_0}(p,p_i) \geq d_{g_0}(p,p_i') - 1/i \tag{7.13}$$
$$d_{g_0}(p_i,q_i) \geq d_{g_0}(p_i',q_i') - 2/i \tag{7.14}$$
$$d_{g_0}(q_i,q) \geq d_{g_0}(q,q_i') - 1/i \tag{7.15}$$

Thus we have our claim because

$$d_{g_i}(p,q) \geq d_{g_0}(p,p_i') + (1/2)d_{g_0}(p_i',q_i') + d_{g_0}(q_i',q) + 3/i \tag{7.16}$$
$$\geq d_\infty(p,q). \tag{7.17}$$

So in fact we have d_i converges pointwise to d_∞. Following the arguments in the first two papers of Allen-Sormani applying the Appendix to Huang-Lee-Sormani and the fact that

$$(1/2)d_{g_0}(p,q) \leq dg_i(p,q) \leq d_{g_0}(p,q) \tag{7.18}$$

we get uniform, intrinsic flat, and Gromov-Hausdorff convergence of

$$(M, d_{g_i}) \to (M, d_\infty) \tag{7.19}$$

which according to (7.5) is not the metric completion of $(M \setminus S, g_0)$ even though $g_i \to g_0$ on compact sets away from S.

Remark 7.2. This example is a counter example to the original statement of Theorem 1.3 because $M_i = (M, g_i)$ is a sequence of compact oriented Riemannian manifolds such that S is a codimension 2 submanifold and we can choose a connected precompact exhaustion,

$$W_j = r^{-1}[2/j, \infty) \subset M \setminus S \tag{7.20}$$

satisfying (1.2)

$$\bar{W}_j \subset W_{j+1} \text{ with } \bigcup_{j=1}^{\infty} W_j = M \setminus S \tag{7.21}$$

with g_i converge smoothly to g_0 on each W_j, in fact $g_i = g_0$ for $i > j$. Furthermore

$$\operatorname{diam}_{M_i}(W_j) \le \operatorname{diam}_{g_0}(M) = D_0 \qquad \forall i \ge j, \tag{7.22}$$

$$\operatorname{Vol}_{g_i}(\partial W_j) \le \operatorname{Vol}_{g_i}(\partial W_j) = 4\pi(2/j)\pi \tag{7.23}$$

and

$$\begin{aligned} \operatorname{Vol}_{g_i}(M \setminus W_j) &\le \operatorname{Vol}_{g_0}(M \setminus W_j) \\ &= (4/3)\pi(2/j)^2\pi = V_j \quad \text{ where } \lim_{j\to\infty} V_j = 0. \end{aligned} \tag{7.24}$$

However

$$\lim_{j\to\infty} d_{\mathcal{F}}(M'_j, N') = 0. \tag{7.25}$$

where N' is the settled completion of $(M \setminus S, g_0)$.

Remark 7.3. This example is not a counter example to Theorem 1.2 because of the highly negative sectional and Ricci curvature near S.

Remark 7.4. This example is not a counter example to Theorem 5.2 because M/S is not uniformly well embedded as defined in the new Definition 5.1. Consider a pair of points $p, q \in M \setminus S$ and $p', q' \in S$ such that

$$d_\infty(p,q) = d_{g_0}(p, p_i) + (1/2)d_{g_0}(p', q') + d_{g_0}(q_i, q) < d_{g_0}(p,q). \tag{7.26}$$

Taking any connected precompact exhaustion W_j of $U = M \setminus S$, we can take $j > k$ sufficiently large that $p, q \in W_j \subset W_k$. We can take i sufficiently large depending on $j > k$ such that

$$W_j \cap r^{-1}[0, 1/(2i_{j,k})] = \emptyset. \tag{7.27}$$

Then

$$\lambda_{i,j,k} = \sup_{x,y \in W_j} |d_{(W_k, g_i)}(x,y) - d_{(M,g_i)}(x,y)| \tag{7.28}$$

$$\geq |d_{(W_k, g_i)}(p,q) - d_{(M,g_i)}(p,q)| \tag{7.29}$$

$$\geq d_{g_0}(p,q) - d_{g_i}(p,q). \tag{7.30}$$

By the pointwise convergence proven in the example we have

$$\limsup_{i\to\infty} \lambda_{i,j,k} = d_{g_0}(p,q) - d_\infty(p,q) \tag{7.31}$$

so

$$\limsup_{j\to\infty} \limsup_{k\to\infty} \limsup_{i\to\infty} \lambda_{i,j,k} \geq d_{g_0}(p,q) - d_\infty(p,q) > 0 \tag{7.32}$$

and we fail to satisfy (5.2).

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