Pacific Journal of Mathematics

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Volume 278 No. 2 December 2015

PACIFIC JOURNAL OF MATHEMATICS
Vol. 278, No. 2, 2015
dx.doi.org/10.2140/pjm.2015.278.447

DIFFERENTIAL HARNACK ESTIMATES FOR POSITIVE SOLUTIONS TO HEAT EQUATION UNDER FINSLER–RICCI FLOW

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We prove first order differential Harnack estimates for positive solutions of the heat equation (in the sense of distributions) under closed Finsler–Ricci flows. We assume suitable Ricci curvature bounds throughout the flow and also assume that the *S*-curvature vanishes along the flow. One of the key tools we use is the Bochner identity for Finsler structures proved by Ohta and Sturm (*Adv. Math.* 252 (2014), 429–448).

1. Introduction

In the past few decades, geometric flows and, more notably among them, the Ricci flow have proved very useful in attacking long standing geometry and topology questions. One important application is finding the so-called round (of constant curvature, Einstein, soliton, etc.) metrics on manifolds by homogenizing a given initial metric.

There is also a hope that similar methods can be applied in the Finsler setting. One might hope to find an answer for, for instance, Professor Chern's question about the existence of Finsler–Einstein metrics on every smooth manifold by using a suitable geometric flow resembling the Ricci flow.

In the Finsler setting, there are notions of Ricci and sectional curvatures, and Bao [2007] has proposed an evolution of Finsler structures that in essence shares a great resemblance with the Ricci flow of Riemannian metrics. The flow Bao suggests is $\partial F^2/\partial t = -2F^2 R$ where $R = (1/F^2)\,\mathrm{Ric}$. In terms of the symmetric metric tensor associated with F and Akbarzadeh's Ricci tensor, this flow takes the form of $\partial g_{ij}/\partial t = -2\,\mathrm{Ric}_{ij}$ which is the familiar Ricci flow.

The notion of Finsler–Ricci flow is very recent and very little has been done about it. Some partial results regarding the existence and uniqueness of such flows

Lakzian is supported by the Hausdorff postdoctoral fellowship at the Hausdorff Center for Mathematics, University of Bonn, Germany.

MSC2010: primary 35K55; secondary 53C21.

Keywords: Finsler–Ricci flow, differential Harnack, gradient estimate, weighted Ricci curvature, heat equation, curvature-dimension.

are obtained in [Azami and Razavi 2013]. Also, the solitons of this flow have been studied in [Bidabad and Yarahmadi 2014]. Our focus in these notes will be to consider a positive solution of the heat equation (in the sense of distributions) under Finsler–Ricci flow and prove first order differential Harnack estimates that are similar to those in the Riemannian case (see [Liu 2009; Sun 2011]). The key tools we use are the Bochner identity for Finsler metrics (pointwise and in the sense of distributions) proven by Ohta and Sturm [2014] and, as is customary in such estimates, the maximum principle.

We should mention that, in this paper, we are not dealing with the existence and Sobolev regularity of such solutions (which is very important and extremely delicate — for example, in the static case, solutions will be C^2 if and only if the structure is Riemannian). For existence and regularity in the static case see [Ohta and Sturm 2009]. Our main theorem is the following.

Theorem 1.1. *Let $(M^n, F(t))$, $t \in [0, T]$ be a closed Finsler–Ricci flow. Suppose there is a real number $K \in \mathbb{R}$ and positive real numbers K_1 and K_2 such that, for all $t \in [0, T]$,*

(i) $-K_1 \leq (\mathrm{Ric}_{ij}(\boldsymbol{v}))_{i,j=1}^n \leq K_2$ *as quadratic forms on T_xM for all $\boldsymbol{v} \in T_xM \setminus \{0\}$, in any coordinate system, $\{\partial/\partial x_i\}$, that is orthonormal with respect to $g_{\boldsymbol{v}}$, and*

(ii) $\boldsymbol{S}$*-curvature vanishes* (*see Section 2.2.7*).

Let $u(x,t) \in L^2([0,T], H^1(M)) \cap H^1([0,T], H^{-1}(M))$ be a positive global solution (*in the sense of distributions*) *of the heat equation under Finsler–Ricci flow*; *i.e., for any test function $\phi \in C^\infty(M)$ and for all $t \in [0, T]$,*

$$\int_M \phi \partial_t u(t,\cdot)\, dm = -\int_M D\phi(\nabla u(t,\cdot))\, dm\, dt. \tag{1}$$

Then, u satisfies

$$F^2(\nabla(\log u)(t,x)) - \theta \partial_t(\log u)(t,x) \leq \frac{n\theta^2}{t} + \frac{n\theta^3 C_1}{\theta - 1} + n^{3/2}\theta^2\sqrt{C_2}, \tag{2}$$

for any $\theta > 1$ and where

$$C_1 = K_1 \quad \text{and} \quad C_2 = \max\{K_1^2, K_2^2\}. \tag{3}$$

Remark 1.2. Our results can be applied to any Finsler–Ricci flow of Berwald metrics on closed manifolds, since the $\boldsymbol{S}$-curvature vanishes for Berwald metrics (for example, see [Ohta 2011]).

We will note that it might be possible to obtain stronger results with fewer curvature bound conditions by using different methods such as Nash–Moser iteration (as is done by Xia [2014] for harmonic functions in the static case).

Integrating the differential Harnack inequalities, in a standard manner, leads to Harnack-type inequalities.

Corollary 1.3. *Let $(M, F(t))$, $t \in [0, T]$ be as in Theorem 1.1. Then for any two points (x, t_1), $(y, t_2) \in M \times (0, T]$ with $t_1 < t_2$, we get*

(4)
$$u(x, t_1) \leq u(y, t_2)\left(\frac{t_2}{t_1}\right)^{2n\epsilon} \exp\left\{\int_0^1 \frac{\epsilon F^2(\gamma'(s))|_\tau}{2(t_2 - t_1)}\, ds + C(n, \epsilon)(t_2 - t_1)(C_1 + \sqrt{C_2})\right\},$$

whenever $\epsilon > 1/2$, and for C depending on n and ϵ only, and where the dependencies of C_1 and C_2 on our parameters are as in Theorem 1.1. Here γ is a curve joining x and y, with $\gamma(1) = x$ and $\gamma(0) = y$, and $F(\gamma'(s))|_\tau$ is the speed of γ at time $\tau = (1 - s)t_2 + st_1$.

The organization of this paper is as follows: in Section 2, we first briefly review some facts and results about differential Harnack estimates in the Riemannian setting and about Finsler geometry; in Section 3, we present lemmas and computations that we need in order to obtain a useful parabolic partial differential inequality; and in Section 4, we will complete the proof of our main theorem.

2. Background

2.1. ***Differential Harnack estimates for heat equations in Riemannian Ricci flow.*** The Ricci flow equation, $\partial g/\partial t = -2\,\mathrm{Ric}$, was first proposed by Richard Hamilton in his seminal paper [1982]. Ricci flow is a heat-type quasilinear partial differential equation but, as is well-known, it enjoys a short-time existence and uniqueness theorem (see [Hamilton 1982]) and has been the key tool in proving the Poincaré and geometrization conjectures.

The gradient estimates for solutions of parabolic equations under Ricci flow are a very important part of Ricci flow theory. Perelman in his groundbreaking work [2002] proves such estimates for the conjugate heat equation; he then benefited from these estimates in the analysis of his $\mathcal{W}$-entropy functional. Since then there have been many important results in this direction (for both heat equation and conjugate heat equation) in, for example, [Kuang and Zhang 2008; Bailesteanu et al. 2010; Cao et al. 2013; Cao and Hamilton 2009; Cao 2008], to name a few.

Since our proof, in spirit, is closer to ones in Liu [2009] and Sun [2011], we will only mention their result without commenting on the other literature in this direction. Their estimates for positive solutions of the heat equation under a closed Ricci flow can be stated as follows.

Theorem [Liu 2009; Sun 2011]. *Let $(M, g(t))$; $t \in [0, T]$ be a closed Ricci flow solution with $-K_1 \leq \mathrm{Ric} \leq K_2$ $(K_1, K_2 > 0)$ along the flow. For $u(x, t)$, a positive solution of the heat equation $(\Delta_{g(t)} - \partial_t)u(x, t) = 0$, one has the first order* *gradient estimate*

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \theta\frac{\partial_t u(x,t)}{u(x,t)} \le \frac{n\theta^2}{t} + \frac{n\theta^3 K_1}{\theta - 1} + n^{\frac{3}{2}}\theta^2(K_1+K_2), \tag{5}$$

where $\theta > 1$.

Their method of proof is to take $f = \log u$ and

$$\alpha := t\left(\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \theta\frac{\partial_t u(x,t)}{u(x,t)}\right) = t(|\nabla f|^2 - \theta\partial_t f) \tag{6}$$

and apply the maximum principle to the parabolic partial differential inequality

$$(\Delta_{g(t)} - \partial_t)\alpha + 2Df(\nabla\alpha) \ge -\frac{\alpha}{t} + \frac{t}{n}(|\nabla f|^2 - \partial_t f)^2 - 2\theta K_1 t|\nabla f|^2 - t\theta^2 n^2(K_1+K_2)^2.$$

This is the method that we will adopt throughout the paper.

2.2. *Finsler structures.*

2.2.1. *Finsler metric.* Let M be a C^∞-connected manifold. A Finsler structure on M consists of a C^∞ Finsler norm $F : TM \to \mathbb{R}$ satisfying the following conditions:

(F1) F is C^∞ on $TM \setminus 0$.

(F2) F restricted to the fibers is positively 1-homogeneous.

(F3) For any nonzero tangent vector $\boldsymbol{y} \in TM$, the approximated symmetric metric tensor defined by

$$g_{\boldsymbol{y}}(\boldsymbol{u}, \boldsymbol{v}) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(\boldsymbol{y} + s\boldsymbol{u} + t\boldsymbol{v})|_{s=t=0} \tag{7}$$

is positive definite.

2.2.2. *Cartan tensor.* One way to measure the nonlinearity of a Finsler structure is to introduce the so-called *Cartan tensor* defined by

$$C_{\boldsymbol{y}} : \otimes^3 TM \to \mathbb{R}, \quad C_{\boldsymbol{y}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \frac{1}{2}\frac{d}{dt}[g_{\boldsymbol{y}+t\boldsymbol{w}}(\boldsymbol{u}, \boldsymbol{v})]. \tag{8}$$

2.2.3. *Legendre transform.* In order to define the *gradient* of a function, we need the Legendre transform, $\mathcal{L}^* : T^*M \to TM$. For $\omega \in T^*M$, let $\mathcal{L}^*(\omega)$ be the unique vector $\boldsymbol{y} \in TM$ such that

$$\omega(\boldsymbol{y}) = F^*(\omega)^2 \quad \text{and} \quad F(\boldsymbol{y}) = F^*(\omega), \tag{9}$$

where F^* is the dual norm to F.

For a smooth function $u : M \to \mathbb{R}$, the gradient of u is $\nabla u(x) := \mathcal{L}^*(Du(x))$.

2.2.4. *Geodesic spray, Chern connection and curvature tensor.* It is easy to see that the geodesic spray in the Finsler setting is of the form $\boldsymbol{G} = y^i \partial/\partial x_i - 2G^i(x, \boldsymbol{y})\partial/\partial y^i$, where

$$G^i(x, \boldsymbol{y}) = \frac{1}{4} g_{\boldsymbol{y}}^{ik} \left\{ 2 \frac{\partial (g_{\boldsymbol{y}})_{jk}}{\partial x_l} - \frac{\partial (g_{\boldsymbol{y}})_{jl}}{\partial x_k} \right\} y^j y^l. \tag{10}$$

The nonlinear connection that we will be using in this work is the Chern connection, the connection coefficients of which are given by

$$\Gamma^i_{jk} = \Gamma^i_{kj} := \frac{1}{2} g^{il} \left\{ \frac{\partial g_{lj}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l} + \frac{\partial g_{kl}}{\partial x_j} - \frac{\partial g_{lj}}{\partial y^r} G^r_k + \frac{\partial g_{jk}}{\partial y^r} G^r_l - \frac{\partial g_{kl}}{\partial y^r} G^r_j \right\}, \tag{11}$$

where $G^i_j := \partial G^i / \partial y^j$ and g is in fact $g_{\boldsymbol{y}}$.

For Berwald metrics, the geodesic coefficients G^i are quadratic in terms of y (by definition) which immensely simplifies the formula for connection coefficients. In fact for Berwald metrics we have $\Gamma^i_{jk} = \partial^2 G^i / \partial y^j \partial y^k$.

Similar to the Riemannian setting, one uses the Chern connection (and the associated covariant differentiation) to define the curvature tensor

$$R^V(X, Y)Z := [\nabla^V_X, \nabla^V_Y]Z - \nabla^V_{[X,Y]}Z, \tag{12}$$

which, of course, depends on a nonzero vector field V.

2.2.5. *Flag and Ricci curvatures.* *Flag curvature* is defined similar to the sectional curvature in the Riemannian setting. For a fixed flag pole $\boldsymbol{v} \in T_xM$ and for $\boldsymbol{w} \in T_xM$, the flag curvature is defined by

$$\mathcal{K}^{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{w}) := \frac{g_{\boldsymbol{v}}(R^{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{w})\boldsymbol{w}, \boldsymbol{v})}{g_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{v}) g_{\boldsymbol{v}}(\boldsymbol{w}, \boldsymbol{w}) - g_{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{w})^2}. \tag{13}$$

The Ricci curvature is then the trace of the flag curvature, i.e.,

$$\operatorname{Ric}(\boldsymbol{v}) := F^2(\boldsymbol{v}) \sum_{i=1}^{n-1} \mathcal{K}^{\boldsymbol{v}}(\boldsymbol{v}, \boldsymbol{e}_i), \tag{14}$$

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1}, \frac{\boldsymbol{v}}{F(\boldsymbol{v})}\}$ constitutes a $g_{\boldsymbol{v}}$-orthonormal basis of T_xM.

2.2.6. *Akbarzadeh's Ricci tensor.* Akbarzadeh's Ricci tensor is defined by

$$\operatorname{Ric}_{ij} := \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{\operatorname{Ric}}{2} \right). \tag{15}$$

It can be shown that the scalar Ricci curvature, Ric, and Akbarzadeh's Ricci tensor, Ric_{ij}, have the same geometrical implications. For further details regarding this tensor, see [Bao and Robles 2004].

2.2.7. ***S**-curvature.* Associated with any Finsler structure, there is one canonical measure, called the Busemann–Hausdorff measure, which is given by

$$dV_F := \sigma_F(x)\, dx_1 \wedge \cdots \wedge dx_n, \tag{16}$$

where $\sigma_F(x)$ is the volume ratio

$$\sigma_F(x) := \frac{\operatorname{vol}(B_{\mathbb{R}^n}(1))}{\operatorname{vol}(\boldsymbol{y} \in T_x M : F(\boldsymbol{y}) < 1)}. \tag{17}$$

The set whose volume appears in the denominator of (17) is called the indicatrix, and there is often no known way to express its volume in terms of F.

The $\boldsymbol{S}$-curvature, which is another measure of nonlinearity, is then defined by

$$\boldsymbol{S}(\boldsymbol{y}) := \frac{\partial G^i}{\partial y^i}(x, \boldsymbol{y}) - y^i \frac{\partial}{\partial x_i}(\ln \sigma_F(x)). \tag{18}$$

For more details, see [Shen 2004], for example.

2.2.8. *Hessian, divergence and Laplacian.* The Hessian in a Finsler structure is defined by

$$\operatorname{Hess}(u)(X, Y) := XY(u) - \nabla_X^{\nabla u} Y(u) = g_{\nabla u}(\nabla_X^{\nabla u} \nabla u, Y). \tag{19}$$

As usual, for a twice differentiable function u,

$$\operatorname{Hess}(u)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial u}{\partial x_k} \tag{20}$$

For a smooth measure $\mu = e^{-\Psi} dx_1 \wedge \cdots \wedge dx_n$ and a vector field V, the divergence is defined by

$$\operatorname{div}_\mu V := \sum_{i=1}^n \left(\frac{\partial V_i}{\partial x_i} - V_i \frac{\partial \Psi}{\partial x_i}\right). \tag{21}$$

Now, using this divergence, one can define the distributional Laplacian of a function $u \in H^1(M)$ by $\Delta u := \operatorname{div}_\mu(\nabla u)$, i.e.,

$$\int_M \phi \Delta u \, d\mu := -\int_M D\phi(\nabla u)\, d\mu, \tag{22}$$

for $\phi \in C^\infty(M)$.

The Finsler distributional Laplacian is nonlinear but fortunately there is a way to relate it to the trace of the Hessian by adding an $\boldsymbol{S}$-curvature term. Indeed, one has

$$\Delta u = \operatorname{tr}_{\nabla u} \operatorname{Hess}(u) - \boldsymbol{S}(\nabla u). \tag{23}$$

For a proof of (23), see for instance [Wu and Xin 2007].

2.3. ***Weighted Ricci curvature and Bochner–Weitzenböck formula.*** The notion of the *weighted Ricci curvature*, Ric_N, of a Finsler structure equipped with a measure μ was introduced by Ohta [2009]. Take a unit vector $\boldsymbol{v} \in T_xM$ and let $\gamma : [-\epsilon, +\epsilon] \to M$ be a short geodesic whose velocity at time $t = 0$ is $\dot\gamma(0) = \boldsymbol{v}$. Decompose the measure μ along γ with respect to the Riemannian volume form; i.e., let $\mu = e^{-\Psi} d\mathrm{vol}_{\dot\gamma}$. Then

$$\mathrm{Ric}_n(\boldsymbol{v}) := \begin{cases} \mathrm{Ric}(\boldsymbol{v}) + (\Psi \circ \gamma)''(0) & \text{if } (\Psi \circ \gamma)'(0) = 0, \\ -\infty & \text{otherwise}, \end{cases} \tag{24}$$

$$\mathrm{Ric}_N(\boldsymbol{v}) := \mathrm{Ric}(\boldsymbol{v}) + (\Psi \circ \gamma)''(0) - \frac{(\Psi \circ \gamma)'(0)^2}{N - n} \quad \text{when } n < N < \infty, \tag{25}$$

$$\mathrm{Ric}(\boldsymbol{v}) := \mathrm{Ric}(\boldsymbol{v}) + (\Psi \circ \gamma)''(0). \tag{26}$$

Also $\mathrm{Ric}_N(\lambda \boldsymbol{v}) := \lambda^2 \, \mathrm{Ric}_N(\boldsymbol{v})$ for $\lambda \geq 0$.

It is proven in [Ohta 2009] that the curvature bound $\mathrm{Ric}_N \geq KF^2$ is equivalent to the Lott–Villani–Sturm $CD(K, N)$ condition.

Using the weighted Ricci curvature bounds, Ohta and Sturm [2014] proved the Bochner–Weitzenböck formulae (both pointwise and integrated versions) for Finsler structures. For $u \in C^\infty(M)$, the pointwise version of the identity and inequality are

$$\Delta^{\nabla u}\left(\frac{F^2(\nabla u)}{2}\right) - D(\Delta u)(\nabla u) = \mathrm{Ric}_\infty(\nabla u) + \|\nabla^2 u\|^2_{HS(\nabla u)} \quad \text{(identity)}, \tag{27}$$

$$\Delta^{\nabla u}\left(\frac{F^2(\nabla u)}{2}\right) - D(\Delta u)(\nabla u) \geq \mathrm{Ric}_N(\nabla u) + \frac{(\Delta u)^2}{N} \quad \text{(inequality)}. \tag{28}$$

3. Estimates

In this section we will gather all the required lemmas and estimates that will be needed to apply the maximum principle.

Evolution of the Legendre transform. Since in the Finsler setting the gradient is nonlinear and depends on the Legendre transform, we will need to know the evolution of the Legendre transform under Finsler–Ricci flow.

Let (M, F) be a Finsler structure evolving under Finsler–Ricci flow. Then the inverse of the Legendre transform is defined by

$$(\mathcal{L}^*)^{-1} : TM \to T^*M, \quad (\mathcal{L}^*)^{-1}(x, y) = (x, p), \quad \text{where } p_i = g_{ij}(x, y) y^j. \tag{29}$$

To explicitly formulate the Legendre transform, we have, for any given $\omega \in T_x^*M$, that $\mathcal{L}^*(\omega) = y \in T_xM$, where y is the unique solution to the *nonlinear* system

$$g(x, y)_{k1} \cdot y^1 + \cdots + g(x, y)_{kn} \cdot y^n = \omega_k, \quad \text{for } k = 1, \ldots, n, \tag{30}$$

or, in the matrix form,

$$g(\boldsymbol{y})\boldsymbol{y} = \omega. \tag{31}$$

Lemma 3.1. *Let $(M, F(t))$ be a Finsler structure evolving by Finsler–Ricci flow. Then the Legendre transform $\mathcal{L}^* : T^*M \to TM$ satisfies*

$$\partial_t \mathcal{L}^* = 2\,\mathrm{Ric}^i_j\,\mathcal{L}^*; \tag{32}$$

i.e., for any fixed 1-form ω with $\mathcal{L}^*(\omega) = \boldsymbol{y} = y^i \partial/\partial x_i \in TM$, we have

$$\partial_t y^i = 2\,\mathrm{Ric}^i_r\, y^r, \tag{33}$$

where $\mathrm{Ric}^i_r := g^{ij}\,\mathrm{Ric}_{jr}$.

Proof. Fix ω and differentiate both sides of (31) with respect to t to get

$$[\partial_t g(\boldsymbol{y})]\boldsymbol{y} + g(\boldsymbol{y})\,\partial_t \boldsymbol{y} = 0. \tag{34}$$

Therefore,

$$\partial_t \boldsymbol{y} = -g(\boldsymbol{y})^{-1}\,\partial_t g(\boldsymbol{y})\boldsymbol{y}. \tag{35}$$

Expanding the right-hand side of (35), we have, for every i,

$$\begin{aligned}\partial_t y^i &= -g(\boldsymbol{y})^{ij}(\partial_t g(\boldsymbol{y}))_{jr} y^r \\ &= 2g(\boldsymbol{y})^{ij}\,\mathrm{Ric}_{jr}(\boldsymbol{y})y^r - g(\boldsymbol{y})^{ij}\left(\frac{\partial g_{jr}}{\partial y^k}\partial_t y^k\right)y^r \\ &= 2\,\mathrm{Ric}^i_r(\boldsymbol{y})y^r.\end{aligned} \tag{36}$$

Notice that the second term in the second line of (36) vanishes by Euler's theorem. □

Evolution of $F^2(\nabla f)$. One crucial step in the proof of the gradient estimates is to be able to estimate the evolution of the term $F^2(\nabla f)$.

Lemma 3.2. *Let $(M, F(t))$ be a time-dependent Finsler structure. Then*

$$\partial_t[F^2(\nabla f)] = 2g^{ij}(Df)[\partial_t f]_i f_j + [\partial_t g^{ij}](Df) f_i f_j. \tag{37}$$

Proof. Simple differentiation gives

$$\begin{aligned}\partial_t[F^2(\nabla f)] &= \partial_t[F^*(Df)^2] \\ &= \partial_t[g^{ij}(Df) f_i f_j] \\ &= 2g^{ij}(Df)[\partial_t f]_i f_j + \partial_t[g^{ij}(Df)] f_i f_j.\end{aligned} \tag{38}$$

Expanding the second term of the last line in (38), we have

$$\partial_t[g^{ij}(Df)] f_i f_j = [\partial_t g^{ij}](Df) f_i f_j + \frac{\partial g^{ij}}{\partial y^k}\,\partial_t y^k(Df) f_i f_j. \tag{39}$$

Using Euler's theorem, the second term of the right-hand side of (39) vanishes. □

Lemma 3.3. *Suppose F is evolving by the Finsler–Ricci flow equation. Then*

$$\partial_t[F^2(\nabla f)] = 2D(\partial_t f)(\nabla f) + 2\,\mathrm{Ric}^{ij}(Df)\, f_i f_j. \tag{40}$$

Proof. It is standard to see that under Finsler–Ricci flow, we have

$$\partial_t g^{ij} = 2\,\mathrm{Ric}^{ij}, \tag{41}$$

where, as before, $\mathrm{Ric}^{ij} := g^{ir} g^{js}\,\mathrm{Ric}_{rs}$. □

4. Proof of main theorem

In this section we will complete the proof of our main theorem. Throughout the rest of these notes, we consider a solution u of the heat equation. The Laplacian, gradient and Legendre transform are all with respect to $V := \nabla u$ and are valid on $M_u := \{x \in M : \nabla u(x) \neq 0\}$.

Let $\sigma(t, x) = t\partial_t f(t, x)$ where $f = \log u$. Then we have $g_{\nabla f} = g_V$. Let

$$\alpha(t, x) := t\{F^2(\nabla f(t, x)) - \theta\,\partial_t f(t, x)\} = tF^2(\nabla f(t, x)) - \theta\sigma. \tag{42}$$

Lemma 4.1. *In the sense of distributions, $\sigma(t, x)$ satisfies the parabolic differential equality*

$$\Delta\sigma - \partial_t\sigma + \frac{\sigma}{t} + 2D\sigma(\nabla f) = t\{-2\,\mathrm{Ric}^{ij}(\nabla f)\, f_i f_j - 2(\mathrm{Ric})^{kl}(\nabla f)\, f_{kl}\}. \tag{43}$$

Proof. We first note that, for any nonnegative test function $\phi \in H^1([0, T] \times M)$ whose support is included in the domain of the local coordinate,

$$\partial_t(D(t\phi)(\nabla f)) = D(\partial_t(t\phi))(\nabla f) + D(t\phi)(\nabla(\partial_t f)) + 2(\mathrm{Ric})^{ij}(\nabla f)\frac{\partial(t\phi)}{\partial x_i}\frac{\partial f}{\partial x_j}. \tag{44}$$

Indeed,

$$\begin{aligned}
\partial_t(D(t\phi)(\nabla f)) \\
&= D(\partial_t(t\phi))(\nabla f) + D(t\phi)(\partial_t(\mathcal{L}^*(Df)) \\
&= D(\partial_t(t\phi))(\nabla f) + D(t\phi)(\partial_t(\mathcal{L}^*)(Df) + \mathcal{L}^*(D\partial_t f)) \\
&= D(\partial_t(t\phi))(\nabla f) + D(t\phi)(\partial_t(\mathcal{L}^*)(Df)) + D(t\phi)(\mathcal{L}^*(D\partial_t f)) \\
&= D(\partial_t(t\phi))(\nabla f) + D(t\phi)(\nabla(\partial_t f)) + 2g^{sj}(\mathrm{Ric})^i_s(\nabla f)\frac{\partial(t\phi)}{\partial x_i}\frac{\partial f}{\partial x_j} \\
&= D(\partial_t(t\phi))(\nabla f) + D(t\phi)(\nabla(\partial_t f)) + 2(\mathrm{Ric})^{ij}(\nabla f)\frac{\partial(t\phi)}{\partial x_i}\frac{\partial f}{\partial x_j}.
\end{aligned} \tag{45}$$

That is,

(46)
$$-D(t\phi)(\nabla(\partial_t f)) = -\partial_t(D(t\phi)(\nabla f)) + D(\partial_t(t\phi))(\nabla f) + 2(\mathrm{Ric})^{ij}(\nabla f)\frac{\partial(t\phi)}{\partial x_i}\frac{\partial f}{\partial x_j}.$$

Multiplying the left-hand side of (43) by ϕ, integrating and then substituting (46), we get

$$(47)\quad \begin{aligned} \boldsymbol{A} &= \int_0^T\int_M \bigl\{-D\phi(\nabla\sigma) + \partial_t\phi\cdot\sigma + \frac{\phi\sigma}{t} + 2\phi D\sigma(\nabla f)\bigr\}\,dm\,dt \\ &= \int_0^T\int_M \bigl\{-D(t\phi)(\nabla(\partial_t f)) + \partial_t(t\phi)\partial_t f + 2t\phi D(\partial_t f)(\nabla f)\bigr\}\,dm\,dt \\ &= \int_0^T\int_M \Bigl\{D(\partial_t(t\phi))(\nabla f) + \partial_t(t\phi)(\Delta f + F^2(\nabla f)) \\ &\qquad + 2(\mathrm{Ric})^{ij}(\nabla f)\frac{\partial(t\phi)}{\partial x_i}\frac{\partial f}{\partial x_j} + 2t\phi D(\partial_t f)(\nabla f)\Bigr\}\,dm\,dt. \end{aligned}$$

Using the estimates we have obtained for $\partial_t[F(\nabla f)^2]$ in Lemmas 3.2 and 3.3, we arrive at

$$(48)\quad \begin{aligned} \boldsymbol{A} &= \int_0^T\int_M \Bigl\{D(\partial_t(t\phi))(\nabla f) + \partial_t(t\phi)(\Delta f) + \partial_t(t\phi)(F^2(\nabla f)) \\ &\qquad +2(\mathrm{Ric})^{ij}(\nabla f)\frac{\partial(t\phi)}{\partial x_i}\frac{\partial f}{\partial x_j} + 2t\phi D(\partial_t f)(\nabla f)\Bigr\}\,dm\,dt \\ &= \int_0^T\int_M \Bigl\{\partial_t(t\phi)(F^2(\nabla f)) + 2(\mathrm{Ric})^{ij}(\nabla f)\frac{\partial(t\phi)}{\partial x_i}\frac{\partial f}{\partial x_j} + t\phi\partial_t[F(\nabla f)^2] \\ &\qquad -2t\phi\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j\Bigr\}\,dm\,dt \\ &= \int_0^T\int_M t\phi\bigl\{-2\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j - 2\,\mathrm{Ric}^{ij}(\nabla f) f_{ij}\bigr\}\,dm\,dt. \end{aligned}$$

Notice that Euler's theorem has been used in the last line of (48). □

Now we can compute a parabolic partial differential inequality for $\alpha(t,x)$ with a similar left-hand side.

Lemma 4.2. *In the sense of distributions, $\alpha(t,x)$ satisfies*

$$(49)\qquad \Delta^V\alpha + 2D\alpha(\nabla f) - \partial_t\alpha + \frac{\alpha}{t} = \boldsymbol{B},$$

where

$$\begin{aligned} \boldsymbol{B} = \theta(2t\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j + 2t\,\mathrm{Ric}^{kl}(\nabla f) f_{kl}) \\ + 2t\,\mathrm{Ric}(\nabla f) + 2t\|\nabla^2 f\|^2_{HS(\nabla f)} - 2t\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j. \end{aligned}$$

Proof. For a nonnegative test function ϕ, one computes

$$\begin{aligned}
(50)\quad & \int_0^T\int_M \Big\{-D\phi(\nabla\alpha)+\partial_t\phi\alpha+\frac{\phi\alpha}{t}+2\phi D\alpha(\nabla f)\Big\}\,dm\,dt \\
&= -\theta \boldsymbol{A}+\int_0^T\int_M \big\{-tD\phi\big(\nabla(F^2(\nabla f))\big)+\partial_t\phi(tF^2(\nabla f)) \\
&\qquad\qquad + \phi(F^2(\nabla f))+2t\phi D(F^2(\nabla f))(\nabla f)\big\}\,dm\,dt \\
&= -\theta \boldsymbol{A}+\int_0^T\int_M \big\{-tD\phi\big(\nabla(F^2(\nabla f))\big)-\phi\cdot\partial_t\big(t(F^2(\nabla f))\big) \\
&\qquad\qquad + \phi(F^2(\nabla f))+2t\phi D(F^2(\nabla f))(\nabla f)\big\}\,dm\,dt \\
&= -\theta \boldsymbol{A}+\int_0^T\int_M \big\{-tD\phi\big(\nabla(F^2(\nabla f))\big)-\phi\cdot\big(F^2(\nabla f)+t\partial_t(F^2(\nabla f))\big) \\
&\qquad\qquad + \phi(F^2(\nabla f))+2t\phi D(F^2(\nabla f))(\nabla f)\big\}\,dm\,dt \\
&= -\theta \boldsymbol{A}+\int_0^T\int_M \big\{-tD\phi\big(\nabla(F^2(\nabla f))\big)-\phi\cdot t\partial_t(F^2(\nabla f)) \\
&\qquad\qquad + 2t\phi D(F^2(\nabla f))(\nabla f)\big\}\,dm\,dt,
\end{aligned}$$

where $\boldsymbol{A}$ is as in (48).

Again using the estimates for $\partial_t[F(\nabla f)^2]$ (as in Lemmas 3.2 and 3.3), we arrive at

$$\begin{aligned}
(51)\quad & \int_0^T\int_M \Big\{-D\phi(\nabla\alpha)+\partial_t\phi\cdot\alpha+\frac{\phi\alpha}{t}+2\phi D\alpha(\nabla f)\Big\}\,dm\,dt \\
&= -\theta \boldsymbol{A}+\int_0^T\int_M \big\{-tD\phi\big(\nabla(F^2(\nabla f))\big)-\phi\cdot t\partial_t(F^2(\nabla f)) \\
&\qquad\qquad + 2t\phi D(F^2(\nabla f))(\nabla f)\big\}\,dm\,dt \\
&= -\theta \boldsymbol{A}+\int_0^T\int_M \big\{-tD\phi\big(\nabla(F^2(\nabla f))\big)-2t\phi D(\partial_t f)(\nabla f) \\
&\qquad\qquad - 2t\phi\,\mathrm{Ric}^{ij}(\nabla f)\,f_i\,f_j+2t\phi D(F^2(\nabla f))(\nabla f)\big\}\,dm\,dt \\
&= -\theta \boldsymbol{A}+\int_0^T\int_M \big\{-tD\phi\big(\nabla(F^2(\nabla f))\big)-2t\phi D(\Delta f)(\nabla f) \\
&\qquad\qquad - 2t\phi D(F^2(\nabla f))(\nabla f)-2t\phi\,\mathrm{Ric}^{ij}\,f_i\,f_j \\
&\qquad\qquad + 2t\phi D(F^2(\nabla f))(\nabla f)\big\}\,dm\,dt \\
&= -\theta \boldsymbol{A}+\int_0^T\int_M \big\{-tD\phi\big(\nabla(F^2(\nabla f))\big) \\
&\qquad\qquad - 2t\phi D(\Delta f)(\nabla f)-2t\phi\,\mathrm{Ric}^{ij}(\nabla f)\,f_i\,f_j\big\}\,dm\,dt.
\end{aligned}$$

By applying the Bochner–Weitzenböck formula (proven in [Ohta and Sturm 2014]; see also Section 2.3) and noticing that $\boldsymbol{S}=0$ implies $\mathrm{Ric}_\infty(\boldsymbol{v})=\mathrm{Ric}(\boldsymbol{v})$, we can continue as follows:

$$\begin{aligned}-\theta\boldsymbol{A}+\int_0^T\int_M\bigl\{-t\,D\phi\bigl(\nabla(F^2(\nabla f))\bigr)-2t\phi D(\Delta f)(\nabla f)-2t\phi\,\mathrm{Ric}^{ij} f_i f_j\bigr\}\,dm\,dt\\ =-\theta\boldsymbol{A}+\int_0^T\int_M\phi\bigl\{2t\,\mathrm{Ric}(\nabla f)+2t\|\nabla^2 f\|^2_{HS(\nabla f)}-2t\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j\bigr\}\,dm\,dt.\end{aligned}$$

Now, substituting $\boldsymbol{A}$ from (47), we have

$$\begin{aligned}\boldsymbol{B}=\theta(2t\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j+2t\,\mathrm{Ric}^{kl}(\nabla f) f_{kl})+2t\,\mathrm{Ric}(\nabla f)\\ +2t\|\nabla^2 f\|^2_{HS(\nabla f)}-2t\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j. \quad\square\end{aligned}$$

Proof of Theorem 1.1. Assume the curvature bounds given in the statement of Theorem 1.1, and assume that the $\boldsymbol{S}$-curvature vanishes. The constants obtained below all depend on our curvature bounds and the ellipticity of the flow.

Let's start with $B(t,x)$:

$$\begin{aligned}B(t,x)=\theta(2t\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j+2t\,\mathrm{Ric}^{kl}(\nabla f) f_{kl})+2t\,\mathrm{Ric}(\nabla f)\\ +2t\|\nabla^2 f\|^2_{HS(\nabla f)}-2t\,\mathrm{Ric}^{ij}(\nabla f) f_i f_j.\end{aligned}$$

Young's inequality tells us that

$$|\mathrm{Ric}^{kl} f_{kl}|\le\frac{\theta}{2}(\mathrm{Ric}^{kl})^2+\frac{1}{2\theta}f_{kl}^2, \tag{52}$$

and therefore

$$2\theta t|\mathrm{Ric}^{kl} f_{kl}|\le t\theta^2(\mathrm{Ric}^{kl})^2+t f_{kl}^2. \tag{53}$$

Pick a normal coordinate system with respect to $g_{\nabla f}$, with $\nabla f(x)=\partial/\partial x_1$ as well as $\Gamma^1_{ij}(\nabla f(x))=0$ for all i, j. Then

$$\mathrm{Ric}^{ij}(\nabla f)=\mathrm{Ric}_{ij}(\nabla f),\quad \|\nabla^2 f\|^2_{HS(\nabla f)}=\sum f_{ij}^2,\quad \sum_{i=1}^n f_{ii}=\Delta f(x), \tag{54}$$

and consequently

$$\begin{aligned}B(t,x)&\ge 2t\theta\,\mathrm{Ric}_{ij}(\nabla f) f_i f_j-t\sum\theta^2(\mathrm{Ric}_{kl})^2-t\sum f_{kl}^2\\ &\qquad+2t\,\mathrm{Ric}(\nabla f)+2t\|\nabla^2 f\|^2_{HS(\nabla f)}-2t\,\mathrm{Ric}_{ij}(\nabla f) f_i f_j\\ &\ge -2t\theta K_1F^2(\nabla f)-2tK_1F^2(\nabla f)+t\sum f_{ij}^2\\ &\qquad-t\theta^2n^2C_2+2tK_1F^2(\nabla f).\end{aligned} \tag{55}$$

On the other hand, one computes

$$\sum f_{ij}^2 \geq \sum f_{ii}^2 \geq \frac{1}{n}\Big(\sum f_{ii}\Big)^2 = \frac{1}{n}(\Delta f)^2. \tag{56}$$

Hence,

$$t\sum f_{ij}^2 \geq \frac{t}{n}(\Delta f)^2. \tag{57}$$

Putting all the above estimates together and noting that $\theta > 1$, we get

$$\begin{aligned} B(t,x) &\geq \frac{t}{n}(\Delta f)^2 - 2t\theta K_1 F^2(\nabla f) - 2tK_1F^2(\nabla f) - t\theta^2 n^2 C_2 + 2tK_1F^2(\nabla f) \\ &\geq \frac{t}{n}(\Delta f)^2 - 2t\theta K_1 F^2(\nabla f) - t\theta^2 n^2 C_2. \end{aligned}$$

Replacing the term Δf with $(F(\nabla f)^2 - \partial_t f)$, we get the inequality

$$B(t,x) \geq \frac{t}{n}(F(\nabla f)^2 - \partial_t f)^2 - 2t\theta C_1 F^2(\nabla f) - t\theta^2 n^2 C_2, \tag{58}$$

where

$$C_1 = K_1, \tag{59}$$

$$C_2 = \max\{K_1^2, K_2^2\}. \tag{60}$$

This means that

$$\begin{aligned} &\Delta^V \alpha + 2D\alpha(\nabla f) - \partial_t \alpha \\ &\qquad \geq -\frac{\alpha}{t} + \frac{t}{n}(F(\nabla f)^2 - \partial_t f)^2 - 2t\theta C_1 F^2(\nabla f) - t\theta^2 n^2 C_2. \end{aligned} \tag{61}$$

This inequality is exactly of the form that appears in [Liu 2009], and a computation similar to the one at the end of the proof of [Liu 2009, Theorem 2] (using the quadratic formula and maximum principle) gives the desired result. For the sake of clarity, we will repeat the computation here.

Let

$$\bar{\alpha} := \alpha - t\frac{n\theta^3 C_1}{(\theta - 1)} - tn^{3/2}\theta^2\sqrt{C_2}. \tag{62}$$

Suppose the maximum of $\bar{\alpha}$ is attained at (x_0, t_0) and suppose $\bar{\alpha}(x_0, t_0) > n\theta^2$ (which implicitly implies $t_0 > 0$). Therefore, at (x_0, t_0), we have

$$0 \geq (\Delta - \partial_t)\bar{\alpha} \geq (\Delta - \partial_t)\alpha. \tag{63}$$

Let $w := F^2(\nabla f)$ and $z := \partial_t f$. Then in terms of w and z we have

$$0 \geq -\frac{\alpha}{t_0} + \frac{t_0}{n}(w - z)^2 - 2t_0\theta C_1 w - t_0\theta^2 n^2 C_2. \tag{64}$$

By the quadratic formula, we get

$$
\begin{aligned}
(65)\quad & \frac{t_0}{n}(w-z)^2 - 2t_0\theta C_1 w \\
&= \frac{t_0}{n}\left(\frac{1}{\theta^2}(w-\theta z)^2 + \left(\frac{\theta-1}{\theta}\right)^2 w^2 - 2\theta n C_1 w + 2\left(\frac{\theta-1}{\theta^2}w\right)(w-\theta z)\right) \\
&\geq \frac{t_0}{n}\left(\frac{1}{\theta^2}(w-\theta z)^2 - \frac{\theta^4 n^2 C_1^2}{(\theta-1)^2} + 2\left(\frac{\theta-1}{\theta^2}w\right)(w-\theta z)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(66)\quad 0 &\geq \frac{t_0}{n\theta^2}\left(\frac{\alpha}{t_0}\right)^2 - \frac{\alpha}{t_0} - \frac{n\theta^4 C_1^2}{(\theta-1)^2}t_0 - t_0\theta^2 n^2 C_2 + \frac{2t_0}{n}\frac{\theta-1}{\theta^2}F^2(\nabla f)\left(\frac{\alpha}{t_0}\right) \\
&\geq \frac{t_0}{n\theta^2}\left(\frac{\alpha}{t_0}\right)^2 - \frac{\alpha}{t_0} - \frac{n\theta^4 C_1^2}{(\theta-1)^2}t_0 - t_0\theta^2 n^2 C_2.
\end{aligned}
$$

Using the quadratic formula one more time, (66) implies that

$$
\frac{\alpha}{t_0} \leq \frac{n\theta^2}{t_0} + \frac{n\theta^3 C_1}{\theta-1} + n^{\frac{3}{2}}\theta^2\sqrt{C_2}, \tag{67}
$$

which in turn implies

$$
\bar{\alpha}(x_0, t_0) \leq n\theta^2, \tag{68}
$$

and this is a contradiction. Therefore,

$$
F^2(\nabla(\log u)(t,x)) - \theta\partial_t(\log u)(t,x) \leq \frac{n\theta^2}{t} + \frac{n\theta^3 C_1}{(\theta-1)} + n^{3/2}\theta^2\sqrt{C_2}, \tag{69}
$$

with C_1 and C_2 as in (59) and (60). □

Proof of Corollary 1.3. From Theorem 1.1, we know that

$$
F^2(\nabla(\log u)(t,x)) - \theta\partial_t(\log u)(t,x) \leq \frac{n\theta^2}{t} + C(n,\theta)(C_1 + \sqrt{C_2}). \tag{70}
$$

Let $l(s) := \ln u(\gamma(s), \tau(s)) = f(\gamma(s), \tau(s))$. Then

$$
\begin{aligned}
(71)\quad \frac{\partial l(s)}{\partial s} &= (t_2 - t_1)\left(\frac{Df(\dot{\gamma}(s))}{t_2 - t_1} - \partial_t f\right) \\
&\leq (t_2 - t_1)\left(\frac{F(\nabla f)F(\dot{\gamma})}{t_2 - t_1} - \partial_t f\right) \\
&\leq (t_2 - t_1)\left(\frac{\epsilon F^2(\dot{\gamma})|_\tau}{2(t_2 - t_1)^2} + \frac{1}{2\epsilon}F^2(\nabla f) - \partial_t f\right) \\
&\leq \frac{\epsilon F^2(\dot{\gamma})|_\tau}{2(t_2 - t_1)} + (t_2 - t_1)\left(\frac{2n\epsilon}{\tau} + C(n,\epsilon)(C_1 + \sqrt{C_2})\right).
\end{aligned}
$$

Integrating this inequality gives

$$\begin{aligned}\ln\frac{u(x,t_1)}{u(y,t_2)} &= \int_0^1 \frac{\partial l(s)}{\partial s}\,ds \\ &\le \int_0^1 \frac{\epsilon F^2(\dot{\gamma})|_\tau}{2(t_2-t_1)}\,ds + C(n,\epsilon)(t_2-t_1)(C_1+\sqrt{C_2}) + 2\epsilon n \ln\frac{t_2}{t_1}. \quad \square\end{aligned}$$

Acknowledgements

The author would like to thank Professor K.-T. Sturm for the opportunity of working under his supervision as a postdoctoral fellow and for his great insights. The author also would like to thank Professor Christina Sormani for her constant support and encouragement. Many thanks go to the stochastic analysis group in Bonn and also my colleagues Mike Munn and Lashi Bandara.

The author is deeply grateful for valuable comments made and numerous corrections suggested by the anonymous reviewer of this paper.

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Received May 8, 2014. Revised March 10, 2015.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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