



Non-branching $RCD(0, N)$ Geodesic Spaces with Small Linear Diameter Growth have Finitely Generated Fundamental Groups

Yu Kitabeppu and Sajjad Lakzian

Abstract. In this paper, we generalize the finite generation result of Sormani to non-branching $RCD(0, N)$ geodesic spaces (and in particular, Alexandrov spaces) with full support measures. This is a special case of the Milnor's Conjecture for complete non-compact $RCD(0, N)$ spaces. One of the key tools we use is the Abresch–Gromoll type excess estimates for non-smooth spaces obtained by Gigli–Mosconi.

1 Introduction

In [17], Milnor conjectures that a complete non-compact Riemannian manifold, M^n , with non-negative Ricci curvature possesses a finitely generated fundamental group. The finite generation of fundamental group has been proved in the following cases:

- (a) If M has non-negative sectional curvature (Cheeger-Gromoll [9]).
- (b) When M is three dimensional and $\text{Ric} > 0$ (Schoen-Yau [22]).
- (c) The complete three dimensional Milnor's conjecture (Liu [15]).
- (d) When M has Euclidean volume growth (Anderson [3] and Li [14]).
- (e) M^n has small diameter growth ($O(r^{\frac{1}{n}})$) and sectional curvature bounded below (Abresch–Gromoll [1]).

Notice that when $n = 2$, the result follows from (a), since $\text{Ric} \geq 0$ is the same as non-negative sectional curvature.

As far as finite generation results in non-smooth spaces satisfying curvature-dimension bounds, Bacher-Sturm in [4] prove the finite generation of the fundamental group for $CD(K, N)$ spaces with $K > 0$. This is a result of the Myer's Theorem in the non-smooth setting.

Sormani [24] proves that a Riemannian manifold M^n ($n \geq 3$) with $\text{Ric} \geq 0$ has a finitely generated fundamental group if it has small linear diameter growth,

Received by the editors November 11, 2014; revised May 28, 2015.

Published electronically July 13, 2015.

Yu Kitabeppu is partly supported by the Grant-in-Aid for JSPS Fellows, The Ministry of Education, Culture, Sports, Science and Technology, Japan, and also partly supported by the grants of Strategic Young Researcher Overseas Visits Program for Accelerating Brain Circulation during his stay in Bonn. Sajjad Lakzian is supported by the Hausdorff post-doctoral fellowship at the Hausdorff Institute for Mathematics, University of Bonn.

AMS subject classification: 53C23, 30L99.

Keywords: Milnor conjecture, non negative Ricci curvature, curvature dimension condition, finitely generated, fundamental group, infinitesimally Hilbertian.

$$\limsup \frac{\text{diam } \partial(B(p, r))}{r} < 4S_n,$$

where the universal constant S_n (coming from the Abresch–Gromoll excess estimate) is

$$S_n := \left(4 \cdot 3^n \left(\frac{n-1}{n-2} \right)^{n-1} \frac{n-1}{n} \right)^{-1}.$$

This result was later extended to smooth metric measure spaces with non-negative Bakery–Emery Ricci curvature (see Wei [27]).

Following the recent progress in the study of metric-measure spaces having curvature bounded from below in the sense of Lott–Sturm–Villani that are also infinitesimally Hilbertian, *i.e.*, having a linear Laplacian (see Gigli [10]), come many tools that were previously only available in the Riemannian setting. Among these tools are the splitting theorem and Abresch–Gromoll excess estimates [12] and [11], to name a few.

Our purpose in this paper is to extend the above mentioned result of Sormani [24] to the spaces satisfying the curvature-dimension condition $CD(0, N)$ that are also infinitesimally Hilbertian (in short, $RCD(0, N)$ spaces). In the course of the proof, it will become clear that we need to assume some other metric conditions on the space, but the general approach is reminiscent of that of [24]. The main theorem of this paper follows.

Theorem 1.1 *Let (X, d_X, m) be a connected, semi-locally simply connected, and non-branching geodesic metric-measure space with $\text{supp}(m) = X$. Suppose X satisfies the $CD(0, N)$ curvature-dimension conditions that is also infinitesimally Hilbertian (see [2] for the detailed definition). If X has small linear diameter growth*

$$\limsup \frac{\text{diam } \partial(B(p, r))}{r} < 4S_N,$$

where

$$S_N = \begin{cases} \left(9 \frac{N-1}{2-N} + 4 \right)^{-1} & \text{if } 1 < N < 2, \\ \frac{1}{13} & \text{if } N = 2, \\ \left(4 + 2 \cdot 3^N \left(\frac{N-1}{N-2} \right)^{N-1} \frac{N-1}{N} \right)^{-1} & \text{if } N > 2, \end{cases}$$

then X has a finitely generated fundamental group.

Since the fundamental group and the diameter growth in nature are independent from the measure on the space, we can rephrase our main theorem in the following different, but more enlightening, manner.

Corollary 1.2 *Let (X, d_X) be a connected, semi-locally simply connected, and non-branching geodesic metric space with small linear diameter growth. If one can find a Borel measure m on X with $\text{supp}(m) = X$ and for which (X, d_X, m) becomes an $RCD(0, N)$ space, then $\pi_1(X)$ is finitely generated.*

Remark 1.3 One might be interested in using the Corollary 1.2 to produce many non-branching examples (and not necessarily coming from Finsler manifolds) of metric spaces that do not possess any Borel measures with full support that would make the space an $RCD(0, N)$ space.

When the underlying space is non-negatively curved in the Alexandrov sense, our result simplifies to the following corollary.

Corollary 1.4 *Let X be a metric space with non-negative curvature in the sense of Alexandrov; then X has a finitely generated fundamental group if X has small diameter growth.*

Remark 1.5 Perelman [19] proved that any non-compact non-negatively curved Alexandrov space X has a closed totally convex subset S , which is a deformation retraction of X . Thus, the fundamental group of X is isomorphic to that of S ; accordingly, it is finitely generated. Corollary 1.4 gives a different proof for Alexandrov spaces with small diameter growth.

This paper is organized as follows. Section 2 is devoted to a brief review of excess estimates in a non-smooth setting. In Section 3, we will discuss universal coverings of $RCD(K, N)$ spaces and their properties. In Section 4, we generalize the half way and uniform cut lemmas to non-smooth spaces, and the proofs of Theorem 1.1 and Corollary 1.4 are presented in Section 5.

2 $RCD(0, N)$ Spaces and Excess Estimates

2.1 Abresch–Gromoll Excess Estimates

Let M be a complete Riemannian manifold. Take two distinct points $y_1, y_2 \in M$ and fix them; then for any $x \in M$, the excess $e(x)$ is

$$e(x) := d(x, y_1) + d(x, y_2) - d(y_1, y_2).$$

It is straightforward that $e(x)$ is a Lipschitz function with Lipschitz constant 2.

Now suppose γ is a minimal geodesic connecting y_1 and y_2 and define the leg and height functions $l(x)$ and $h(x)$ (resp.) as

$$l(x) := \min\{d(x, y_1), d(x, y_2)\} \quad \text{and} \quad h(x) := \min_t d(x, \gamma(t)).$$

The triangle inequality implies $e(x) \leq 2h(x)$.

The significance of the Abresch–Gromoll excess estimate is that they give a non-trivial upper bound for the excess that has the right asymptotic behavior.

Abresch and Gromoll [1] prove that when $\text{Ric} \geq 0$ and when $h(x) \leq \frac{l(x)}{2}$,

$$e(x) \leq 4 \left(\frac{h^n(x)}{l(x)} \right)^{1/(n-1)}.$$

2.2 RCD(0,N) Spaces

Sturm in [25, 26] and Lott and Villani in [16] independently developed a notion of a metric measure space having Ricci curvature being bounded from below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1, \infty]$. The conditions that these spaces must satisfy are called the *curvature-dimension conditions*, and these spaces are said to be of class $CD(K, N)$.

These curvature-dimension bounds actually generalize the smooth Ricci curvature bounds for Riemannian manifolds. Another nice property of the $CD(K, N)$ classes is their closedness under measured Gromov–Hausdorff convergence (cf. [16]).

In order to get a “local-to-global” property on top of the aforementioned properties, Bacher and Sturm [4] introduce a variation of the curvature-dimension conditions that is called the reduced curvature-dimension condition, *i.e.*, $CD^*(K, N)$.

The $CD^*(K, N)$ condition, while being a local condition, can be realized by some non-linear Finsler structures. It is well known (see Cheeger–Colding [7, 8]) that non-linear Finsler structures do not arise as limits of Riemannian structures with Ricci curvature bounded below and they can exhibit undesirable behaviour that would not match with our expectation of a space with curvature bounded below.

To exclude these non-linear anomalies, Ambrosio–Gigli–Savare [2] define the notion of a space being “infinitesimally Hilbertian” spaces. To wit, *infinitesimally Hilbertian* means that the space enjoys a linear Laplacian or equivalently the Sobolev space $W^{1,2}(X, d_X, m)$ is Hilbert. An $RCD(K, N)$ space is a $CD(K, N)$ space that is also infinitesimally Hilbertian.

The $RCD(K, N)$ condition is again stable under measured Gromov–Hausdorff convergence and is also compatible with the smooth Riemannian setting.

A key property that we will benefit from in our arguments is that $CD^*(0, N) = CD(0, N)$. See [4] for a proof. This means that $RCD(0, N)$ spaces (or equivalently $RCD^*(K, N)$ spaces) enjoy a nice local to global property.

2.3 Excess Estimates for RCD(K,N) Spaces

Gigli and Mosconi [12] prove Abresch–Gromoll type excess estimates for $RCD(0, N)$ spaces. They also generalize Cheeger and Colding’s excess estimates that appeared in [6]. For the sake of clarity, we will outline Gigli and Mosconi’s result in below.

Let (X, d_X, m) be an $RCD(K, N)$ space for some $K \leq 0$ and for $1 < N < \infty$. Let $\bar{x} \in \text{supp}(m)$ be a point in the support of the background measure. Furthermore, assume that the leg and height functions satisfy $h(\bar{x}) < l(\bar{x})$; then

$$(2.1) \quad e(\bar{x}) \leq \begin{cases} 2^{\frac{N-1}{N-2}} (D_{K,N}(\bar{x})h^N(\bar{x}))^{\frac{1}{N-1}} & \text{if } N > 2, \\ \frac{N}{2-N} D_{K,N}(\bar{x})h^2(\bar{x}) & \text{if } 1 < N < 2, \\ D_{K,N}(\bar{x})h^2(\bar{x}) \left(\frac{1}{1+\sqrt{1+D^2(\bar{x})h^2(\bar{x})}} + \log \frac{1+\sqrt{1+D^2(\bar{x})h^2(\bar{x})}}{D_{K,N}(\bar{x})h(\bar{x})} \right) & \text{if } N = 2, \end{cases}$$

where

$$D_{K,N}(\tilde{x}) = \left(\frac{s_{K,N}(h(\tilde{x}))}{h(\tilde{x})} \right)^{N-1} \frac{c_{K,N}(l(\tilde{x}) - h(\tilde{x}))}{N},$$

$$s_{K,N}(\theta) = \begin{cases} \sqrt{\frac{N-1}{K}} \sin\left(\theta\sqrt{\frac{K}{N-1}}\right) & \text{if } K > 0, \\ \theta & \text{if } K = 0, \\ \sqrt{\frac{N-1}{-K}} \sinh\left(\theta\sqrt{\frac{-K}{N-1}}\right) & \text{if } K < 0, \end{cases}$$

and

$$c_{K,N}(\theta) = \begin{cases} \frac{N-1}{\theta} & \text{if } K = 0, \\ \sqrt{-K(N-1)} \operatorname{coth}\left(\theta\sqrt{\frac{-K}{N-1}}\right) & \text{if } K < 0. \end{cases}$$

When $K = 0$, these estimates simplify to

$$e(\tilde{x}) \leq \begin{cases} \frac{N-1}{2-N} \frac{h^2(\tilde{x})}{l(\tilde{x})-h(\tilde{x})} & \text{if } 1 < N < 2, \\ \frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x})-h(\tilde{x})} \left(\frac{1}{1+\sqrt{1+\left(\frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x})-h(\tilde{x})}\right)^2}} + \log \frac{1+\sqrt{1+\left(\frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x})-h(\tilde{x})}\right)^2}}{\frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x})-h(\tilde{x})}} \right) & \text{if } N = 2, \\ 2 \frac{N-1}{N-2} \left(\frac{N-1}{N} \frac{h^N(\tilde{x})}{l(\tilde{x})-h(\tilde{x})} \right)^{\frac{1}{N-1}} & \text{if } N > 2. \end{cases}$$

3 Universal Covers of $RCD(0, N)$ Spaces

In this section, we will discuss the properties and natural metric measure structure of the universal cover of an $RCD(0, N)$ space. Let X be a topological space; then a covering $P: \tilde{X} \rightarrow X$ is called the *universal cover* if \tilde{X} is simply connected. It is well known that any other covering of X can itself be covered by the universal cover.

For existence of the universal cover we only need to require very mild topological conditions. In fact, if X is connected, locally path-wise connected and semi-locally simply connected, then a universal cover of X exists (see [4] for details).

In this paper we will need to be able to apply the excess estimates (2.1) (see also [12]) to a universal covering of an $RCD(0, N)$ metric measure space X . Hence, we will need a canonical metric measure structure on a universal covering of a metric measure space X .

Let (X, d_X, m) be a metric measure space and let $P: \tilde{X} \rightarrow X$ be a universal covering.

Canonical Metric, $\tilde{d}_{\tilde{X}}$, on \tilde{X}

For a pair of points $\tilde{x}, \tilde{y} \in \tilde{X}$, the metric $\tilde{d}_{\tilde{X}}$ (distance) is defined as

$$\tilde{d}_{\tilde{X}}(\tilde{x}, \tilde{y}) := \inf \{ \operatorname{Length}(\tilde{\gamma}) \mid \tilde{\gamma} \text{ is continuous and connects } \tilde{x} \text{ to } \tilde{y} \}.$$

Notice that $\operatorname{Length}(\tilde{\gamma})$ is computed using the length structure of the base space X and the fact that \tilde{X} is locally homeomorphic to X . The covering map $P: (\tilde{X}, \tilde{d}_{\tilde{X}}) \rightarrow (X, d)$ becomes a local isometry and 1-Lipschitz map.

Remark 3.1 Notice that if \tilde{x} and \tilde{y} are two different points in \tilde{X} , that both project to $p \in X$, Then $\tilde{d}_{\tilde{X}}(\tilde{x}, \tilde{y}) > 0$, while obviously $d_X(p, p) = 0$. This is due to the fact that \tilde{x} and \tilde{y} are on different *sheets* of the covering. To make this more clear, suppose $\tilde{\gamma}$ is a curve joining \tilde{x} and \tilde{y} . Then $\gamma := P \circ \tilde{\gamma}$ is a closed curve whose image does not entirely lie inside $B_{r_p}(p)$ for some $r_p > 0$ small enough (dues to the definition of a covering space). This means that

$$\text{Length}(\tilde{\gamma}) \geq \text{Length}(\gamma) > 2r_p > 0.$$

Canonical Measure, \tilde{m} , on \tilde{X}

Again using the properties of a covering map, one can canonically obtain a measure \tilde{m} on the covering space, \tilde{X} . Let $\tilde{A} \subset \tilde{X}$ be any subset such that the restriction of the covering map P to \tilde{A} is an isometry to $P(\tilde{A})$. Define $\tilde{m}(\tilde{A}) := m(P(\tilde{A}))$ and then extend this measure to the σ -algebra generated by all such sets, which in turn is equal to the Borel σ -algebra of \tilde{X} (for details see [4]).

The measure \tilde{m} can also be defined in the following equivalent manner:

$$\tilde{m}(\tilde{U}) := \sup \left\{ \sum m(P(\tilde{A}_j)) \mid \tilde{U} = \sqcup \tilde{A}_j \right\}.$$

Theorem 3.2 Suppose that X is a non-branching geodesic space and assume that the universal cover \tilde{X} exists. Then $(\tilde{X}, \tilde{d}_{\tilde{X}}, \tilde{m})$ is an $RCD(0, N)$ space whenever (X, d_X, m) is an $RCD(0, N)$ space.

Proof Since (X, d, m) is an $RCD(0, N)$ space, it is also an $RCD^*(0, N)$ space. Namely, it is infinitesimally Hilbertian and a $CD^*(0, N)$ space. It is known that in non-branching spaces, the $RCD^*(0, N)$ condition is a local property. Also it is obvious that \tilde{X} is also non-branching. Now, by the construction of $\tilde{d}_{\tilde{X}}$ and \tilde{m} , $(\tilde{X}, \tilde{d}_{\tilde{X}}, \tilde{m})$ is also an $RCD^*(0, N)$ space. Since $CD^*(0, N)$ and $CD(0, N)$ are equivalent to each other, $(\tilde{X}, \tilde{d}_{\tilde{X}}, \tilde{m})$ is an $RCD(0, N)$ space. ■

4 Half Way Lemma and Uniform Cut Lemma in a Non-Smooth Setting

4.1 Half Way Lemma

From now on, we always assume that a metric space (X, d) is connected, locally path-wise connected, and semi-locally simply connected. Accordingly the existence of the universal cover is guaranteed.

Lemma 4.1 (Halfway Lemma) Let (X, d) be a connected and geodesic metric space. Assume furthermore that X is proper. Then there exists an ordered set of independent generators $\{g_1, g_2, \dots\}$ having minimal representative geodesic loops γ_k with $\text{Length}(\gamma_k) = d_k$ such that

$$(4.1) \quad d_X\left(\gamma_k(0), \gamma_k\left(\frac{d_k}{2}\right)\right) = \frac{d_k}{2},$$

and if $\pi_1(X, x_0)$ is infinitely generated, one obtains a sequence of such generators.

Proof First we note that \tilde{X} is proper if and only if so is X . Now fix $x_0 \in X$ and let $\tilde{x}_0 \in \tilde{X}$ be a lift of x_0 to \tilde{X} . Obviously, for any non-trivial element $g \in G = \pi_1(X, x_0)$, one has $\tilde{d}_{\tilde{X}}(\tilde{x}_0, g\tilde{x}_0) > 0$. Furthermore, the semi-locally simply connectedness and the properness of X guarantee the positivity of a minimal value of $\tilde{d}_{\tilde{X}}(\tilde{x}_0, g\tilde{x}_0)$. Indeed, suppose there exists a sequence $\{g_i\} \subset \pi_1(X, x_0)$ such that $\tilde{d}_{\tilde{X}}(\tilde{x}_0, g_i\tilde{x}_0) \rightarrow 0$. This means that for any neighbourhood U of x_0 , there exists a large number J such that $g_j \in \pi_1(U, x_0)$ for any $j \geq J$. Since each g_j is not a trivial element even in $\pi_1(X, x_0)$, the homomorphism induced by the inclusion map $\iota: U \rightarrow X$ is not trivial. This contradicts the definition of semi-locally simply connected.

Since G is discrete, there exists an element $g_1 \in G$ attaining the minimum. Now we can proceed by induction just as in [24], and we obtain (4.1) as in [24]. ■

Remark 4.2 An $RCD(0, N)$ space X is proper. However, it is not known to the authors whether an $RCD(0, N)$ is also automatically semi-locally simply connected or not.

4.2 Uniform Cut Lemma

To generalize the uniform cut lemma of [24] to our setting, we need some non-branching assumptions on $RCD(0, N)$ space X . Moreover, we need to modify the value of S_N .

Define the universal constant S_N by

$$(4.2) \quad S_N = \begin{cases} \left(9 \frac{N-1}{2-N} + 4\right)^{-1} & \text{if } 1 < N < 2, \\ 1/13 & \text{if } N = 2, \\ \left(4 + 2 \cdot 3^N \left(\frac{N-1}{N-2}\right)^{N-1} \frac{N-1}{N}\right)^{-1} & \text{if } N > 2. \end{cases}$$

We define the *set of cut locus* for $x \in X$, roughly-speaking, as the set of points at which geodesics emanating from x stop being minimizing.

Definition 4.3 (Cut Locus) Let (X, d) be a complete geodesic metric space. For a given point $x \in X$, we define the set of cut locus at x by

$$C_x := \left\{ y \in X; \exists z \in X \text{ s.t. } d(x, z) = d(x, y) + d(y, z) \right\}.$$

An element in C_x is called a *cut point*. For Riemannian manifolds, they coincide with the ordinal cut locus.

Remark 4.4 Since our space is a geodesic space, it is straightforward to see that the notion of cut locus as defined here coincides with the minimal cut locus as defined in [23]. The interested reader should consult [23] for a thorough discussion of conjugate and cut points in length spaces.

Lemma 4.5 (Uniform Cut Lemma for $N \neq 2$) Let (X, d, m) be a non-branching $RCD(0, N)$ space with $N \neq 2$. Let γ be a non-contractible geodesic loop based at a point $x_0 \in X$ with $\text{Length}(\gamma) = L$. Suppose the following are true:

- (i) γ has the shortest length among all loops homotopic to γ ;
- (ii) γ is minimal on both intervals $[0, \frac{L}{2}]$ and $[\frac{L}{2}, L]$.

Then for any $x \in \partial B(x_0, RL)$ with $L \geq \frac{1}{2} + S_N$, one has

$$d_X\left(x, \gamma\left(\frac{L}{2}\right)\right) \geq \left(R - \frac{1}{2}\right)L + 2S_N L,$$

where S_N is the universal constant defined by (4.2).

For the sake of completeness, we outline the proof (which is similar to the one in [24]) below.

Proof Throughout the proof we have $N \neq 2$. We first observe that for a geodesic loop $\gamma: [0, L] \rightarrow X$ based at a point x_0 , There does not exist geodesic from x_0 through $\gamma(L/2)$ such that it is still minimal after passing through $\gamma(L/2)$. On the contrary, suppose that there exists such geodesic $\eta: [0, L/2 + \epsilon] \rightarrow X$. Both curves $\eta(L/2 \rightarrow L/2 + \epsilon) \circ \gamma(0 \rightarrow L/2)$ and $\eta(L/2 \rightarrow L/2 + \epsilon) \circ \gamma(L \rightarrow L/2)$ are minimal geodesics and this contradicts the non-branching property of X . The above claim means that $\gamma(L/2) \in C_{x_0}$ and $d(x, \gamma(L/2)) > L/2 + RL$ for any $x \in \partial B(x_0, RL)$, $R > 1/2$.

For $R_0 = \frac{1}{2} + L$, we will examine the proof of a uniform cut lemma in our setting. Suppose there exists a point $x \in \partial B(x_0, R_0L)$ with

$$d_X\left(x, \gamma\left(\frac{L}{2}\right)\right) =: A < 3S_N L.$$

Let $\beta: [0, A] \rightarrow X$ be a minimal geodesic from $\gamma(\frac{L}{2})$ to x . Consider the triangle in \tilde{X} with vertices $\tilde{x}_0, g\tilde{x}_0$, and \tilde{x} and with geodesic legs given by the lifts $\tilde{\gamma}$ from \tilde{x}_0 to $g\tilde{x}_0$ and $\tilde{\beta}(0 \rightarrow A) \circ \tilde{\gamma}(0 \rightarrow \frac{L}{2})$ from \tilde{x}_0 to \tilde{x} . Let

$$\begin{aligned} \tilde{l}_0 &:= \tilde{d}_{\tilde{X}}(\tilde{x}, \tilde{x}_0) \geq d_X(x, x_0) = R_0L, \\ \tilde{l}_1 &:= \tilde{d}_{\tilde{X}}(\tilde{x}, g\tilde{x}_0) \geq d_X(x, x_0) = R_0L. \end{aligned}$$

Now on one hand, the excess at \tilde{x} satisfies

$$e(\tilde{x}) := \tilde{l}_0 + \tilde{l}_1 - \tilde{d}_{\tilde{X}}(\tilde{x}_0, g\tilde{x}_0) \geq 2R_0L - L = 2S_N L,$$

so we can apply the non-smooth excess estimates. And on the other hand, since $S_N < \frac{1}{8}$, one observes that

$$l(\tilde{x}) - h(\tilde{x}) \geq \left(\frac{1}{2} + S_N\right)L - 3S_N L = L\left(\frac{1}{2} - 2S_N\right) > \frac{L}{4}.$$

Now, applying the Abresch–Gromoll type excess estimates for $RCD(0, N)$ spaces (see [12]) yields

$$2S_N L \leq e(\tilde{x}) < \begin{cases} \frac{N-1}{2-N} \frac{(3S_N L)^2}{L(\frac{1}{2}-2S_N)} & \text{if } 1 < N < 2, \\ 2 \frac{N-1}{N-2} \left(\frac{N-1}{N} \frac{(3S_N L)^N}{L(\frac{1}{2}-2S_N)}\right)^{\frac{1}{N-1}} & \text{if } N > 2. \end{cases}$$

The above inequalities simplify to

$$\begin{cases} S_N > \left(9 \frac{N-1}{2-N} + 4\right)^{-1} & \text{if } 1 < N < 2, \\ S_N > \left(2 \cdot 3^N \frac{N-1}{N} \left(\frac{N-1}{N-2}\right)^{N-1} + 4\right)^{-1} & \text{if } N > 2. \end{cases}$$

Both inequalities contradict the definition of S_N .

For $R \geq R_0$, take $y \in \partial B(x_0, R_0L) \cap \gamma(0 \rightarrow \frac{L}{2})$ and compute

$$\begin{aligned} d_X\left(x, \gamma\left(\frac{L}{2}\right)\right) &= d_X(x, y) + d_X\left(y, \gamma\left(\frac{L}{2}\right)\right) \\ &\geq (RL - R_0L) + 3S_NL = \left(R - \frac{1}{2}\right)L + 2S_NL. \quad \blacksquare \end{aligned}$$

Lemma 4.6 (Uniform Cut Lemma for $N = 2$) *Let (X, d, m) be a non-branching $RCD(0, 2)$ space. Let γ be a non-contractible geodesic loop based at a point $x_0 \in X$ with $\text{Length}(\gamma) = L \geq 1352$. Suppose the following are true:*

- (i) γ has the shortest length among all loops homotopic to γ .
- (ii) γ is minimal on both intervals $[0, \frac{L}{2}]$ and $[\frac{L}{2}, L]$.

Then for any $x \in \partial B(x_0, RL)$ with $R \geq \frac{1}{2} + S_2$, one has

$$d_X\left(x, \gamma\left(\frac{L}{2}\right)\right) \geq \left(R - \frac{1}{2}\right)L + 2S_2L,$$

where S_2 is the universal constant defined in (4.2).

Proof In the same way as in the proof of Lemma 4.5, we have

$$\begin{aligned} (4.3) \quad &2S_2L \\ &\leq e(\tilde{x}) \\ &\leq \frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})} \left(\frac{1}{1 + \sqrt{1 + \left(\frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})}\right)^2}} + \log \frac{1 + \sqrt{1 + \left(\frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})}\right)^2}}{\frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})}} \right) \\ &\leq \frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})} \left(\frac{1}{2} + \frac{2 + \frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})}}{\frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})}} \right) \\ &= \frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})} \left(\frac{3}{2} + 4 \frac{l(\tilde{x}) - h(\tilde{x})}{h^2(\tilde{x})} \right). \end{aligned}$$

Let $t_0 \in [0, 1]$ be a point satisfying $h(\tilde{x}) = \tilde{d}(\tilde{x}, \tilde{\gamma}(t_0L))$. Without loss of generality, we can assume that $t_0 \in [0, 1/2]$. Then one can bound $h(\tilde{x})$ from below as

$$h(\tilde{x}) \geq d(x, \gamma(t_0L)) \geq d(x, \gamma(0)) - d(\gamma(0), \gamma(t_0L)) \geq S_2L.$$

On the other hand, we have

$$l(\tilde{x}) - h(\tilde{x}) \leq \tilde{d}(\tilde{x}, g\tilde{x}_0) - \tilde{d}(\tilde{x}, \tilde{\gamma}(t_0L)) \leq \tilde{d}(\tilde{\gamma}(L), \tilde{\gamma}(t_0L)) = (1 - t_0)L;$$

therefore,

$$(4.4) \quad \frac{l(\tilde{x}) - h(\tilde{x})}{h^2(\tilde{x})} \leq \frac{(1 - t_0)L}{S_2^2L^2} \leq \frac{1}{S_2^2L} \leq \frac{1}{\frac{1}{169} \cdot 1352} = \frac{1}{8}.$$

Combining the two inequalities (4.3) and (4.4), one obtains

$$2S_2L \leq \frac{1}{2} \frac{h^2(\tilde{x})}{l(\tilde{x}) - h(\tilde{x})} \left(\frac{3}{2} + 4 \frac{l(\tilde{x}) - h(\tilde{x})}{h^2(\tilde{x})} \right) < \frac{1}{2} \frac{(3S_2L)^2}{L(\frac{1}{2} - 2S_2)} \left(\frac{3}{2} + 4 \cdot \frac{1}{8} \right) = 2 \frac{9S_2^2L}{1 - 4S_2},$$

or

$$1 < \frac{9S_2}{1 - 4S_2}.$$

This is a contradiction to the definition of S_2 . ■

Remark 4.7 As we see in Lemma 4.6, when $N = 2$, the uniform cut lemma holds only for loops whose lengths are sufficiently large ($L > 1352$). But this is enough for us since in the proof of Theorem 1.1 we have a sequence of loops, the lengths of which are diverging to ∞ .

Remark 4.8 It is known that an $RCD(0, N)$ spaces are strongly $CD(0, N)$, which implies that they are essentially non-branching (see [21] for details), but this is not strong enough to get a topological result as in these notes.

5 Small Diameter Theorem

In this section we prove Theorem 1.1 and Corollary 1.4

Proof of Theorem 1.1 Now that we have all the essential ingredients (Half Way and Uniform Cut Lemmas), the proof the main theorem is essentially the same as the proof of the small diameter growth theorem in [24]. For the sake of completeness, we will repeat the proof below.

Suppose $\pi_1(X, x_0)$ is infinitely generated. Construct the ordered set of independent generators g_1, g_2, \dots as in Lemma 4.1 with minimal representative loops $\gamma_1, \gamma_2, \dots$ (resp.).

First observe that $d_k := \text{Length}(\gamma_k)$ diverges to infinity, since otherwise we would have, for some large R , $\pi_1(X, x_0) = \pi_1(\overline{B(x_0, R)}, x_0)$, which is finite (since $\overline{B(x_0, R)}$ is compact), which is a contradiction.

Let $\{x_k\}$ be a sequence with $x_k \in \partial B(x_0, (\frac{1}{2} + S_N)d_k)$ and let $\beta_k: I \rightarrow X$ be the minimal geodesic from x_k to x_0 . From the uniform cut lemma (Lemma 4.5) we have

$$d_X\left(x_k, \gamma_k\left(\frac{d_k}{2}\right)\right) \geq 3S_N d_k.$$

Now take the points $y_k \in \partial B(x_0, \frac{d_k}{2}) \cap \beta_k(I)$; then, by the triangle inequality, we get

$$d_X\left(y_k, \gamma_k\left(\frac{d_k}{2}\right)\right) \geq 2S_N d_k.$$

Hence,

$$\limsup \frac{\text{diam } \partial(B(p, r))}{r} \geq \limsup \frac{d_X\left(y_k, \gamma_k\left(\frac{d_k}{2}\right)\right)}{\frac{d_k}{2}} \geq 4S_N,$$

which is a contradiction. ■

Proof of Corollary 1.4 Let X be a metric space with non-negative curvature in the sense of Alexandrov and with small linear diameter growth. It is well known that X is non-branching (for example see [5]). From Petrunin [20], we know that X is a $CD(0, N)$ space and infinitesimal Hilbertianity follows from Kuwae–Machigashira–Shioya [13]. The local (Lipschitz) contractibility of X follows from Mitsuishi–Yamaguchi [18] and Perelman [19]. So X satisfies all the hypotheses of Theorem 1.1. ■

Example 5.1 Let (M_i, d_i, ν_i, p_i) be a sequence of pointed complete N -dimensional Riemannian manifolds with nonnegative Ricci curvature. Assume that there exists a positive (small) constant $\epsilon > 0$ such that for any $\eta > 0$, there exists a large number $R > 0$, possibly depending on η , such that inequalities

$$\frac{\text{diam } \partial(B(p_i, r))}{r} < 4S_N - \epsilon + \eta$$

hold for any $i \in \mathbb{N}$, any $r > R$, and furthermore (M_i, d_i, ν_i, p_i) converges to a pointed metric measure space (X, d, m, p) in the pointed Gromov-Hausdorff sense. Suppose (X, d, m) is non-branching; then X has a finitely generated fundamental group. The proof is quite simple, but for the reader's convenience, we give a rough proof below. By the stability result for $RCD(0, N)$ condition, (X, d, m) is also an $RCD(0, N)$ space. Let $\eta > 0$ be a positive constant with $\eta < \epsilon/4$. For a given $R > 0$, we are able to find a large number J such that for any points $x, y \in B_R(p)$ there exist points $x_j, y_j \in B_R(p_j)$ with $|d(x, y) - d(x_j, y_j)| \leq \eta$ for any $j \geq J$. This implies that

$$\limsup_{r \rightarrow \infty} \frac{\text{diam } \partial B(p, r)}{r} < 4S_N.$$

Acknowledgments The authors would like to thank professor Karl Theodor Sturm and the stochastic analysis group at the University of Bonn. Sajjad Lakzian would also like to thank professor Christina Sormani for teaching the authors about this result in the smooth Riemannian setting.

The authors are also very grateful to the anonymous reviewer of this paper for their valuable comments and corrections.

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