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When the rough metric induces a metric measure space satisfying a Riemannian curvature dimension condition, we demonstrate that the distance induced by the flow is identical to the evolving distance metric defined by Gigli and Mantegazza on appropriate admissible points. Consequently, we demonstrate that a smooth compact manifold with a finite number of geometric conical singularities remains a smooth manifold with a smooth metric away from the cone points for all future times. Moreover, we show that the distance induced by the evolving metric tensor agrees with the flow of $\mathrm{RCD}(K, N)$ spaces defined by Gigli and Mantegazza.

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CONTENTS

1. INTRODUCTION

Nearly ten years ago, using the tools of optimal transportation, Lott-Villani [19] and Sturm [27, 28] extended the notion of lower Ricci curvature bounds to the setting of general metric measure spaces. Among other things, they showed that this so-called curvature-dimension condition, denoted $\mathrm{CD}(K, N)$ for $K \in \mathbb{R}$, $N \in [1, \infty]$, is stable under measured Gromov-Hausdorff limits and consistent with the notion of Ricci curvature lower bounds for Riemannian manifolds. That is to say, for smooth

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Riemannian manifolds, the $\mathrm{CD}(K,N)$ condition is equivalent to having the Ricci curvature tensor bounded below by K and dimension of the manifold at most N. In a similar way, for a metric measure space $(\mathcal{X},\mathrm{d},\mu)$, the $\mathrm{CD}(K,N)$ condition is understood to say that $\mathcal{X}$ has N-dimensional Ricci curvature bounded below by K.

Although $\mathrm{CD}(K,N)$ spaces enjoy many favourable properties, Villani shows in [29] that such spaces also allow for Finsler structures. This is a somewhat unsettling fact as it is known that Finsler manifolds cannot arise as the Gromov-Hausdorff limit of Riemannian manifolds with lower Ricci curvature bounds. Even more so, classical results such as the Cheeger-Gromoll splitting theorem were known to fail for general metric measure spaces which are merely $\mathrm{CD}(K,N)$. In order to retain these nice properties while also ruling out Finsler geometries, Ambrosio-Gigli-Savaré introduced a further refined version of the curvature-dimension bound requiring that in addition, the Sobolev space $\mathrm{W}^{1,2}(\mathcal{X})$ is a Hilbert space. Combining this condition of *infinitesimally Hilbertian* structure with the classical curvature dimension condition, they define the *Riemannian Curvature Dimension* condition, denoted $\mathrm{RCD}(K,N)$.

In recent years there has been an increased interest in better understanding the fine geometric and analytic consequences of this Riemannian curvature dimension condition. There has been a great deal of progress in this direction and a number of very deep results describing the structure of these spaces. See, for example, recent work of Ambrosio, Cavalletti, Gigli, Mondino, Naber, Rajala, Savaré, Sturm in [2, 3, 8, 13, 11, 14, 21]. We emphasise that this list is by no means exhaustive, and encourage the reader to consult the references within.

The starting point of our considerations is the paper [13] by Gigli and Mantegazza, where they define a geometric flow for spaces that are possibly singular. There, the authors consider a compact $\mathrm{RCD}(K,N)$ space $(\mathcal{X},\mathrm{d},\mu)$ and define a family of evolving distance metrics d_t for positive time. They build this via the heat flow of d and μ in *Wasserstein space*, the space of probability measures on $\mathcal{X}$ with the so-called Wasserstein metric. The essential feature of this flow is when the triple $(\mathcal{X},\mathrm{d},\mu)$ arises from a smooth compact manifold $(\mathcal{M},\mathrm{g})$. In this setting, the evolution d_t is given by an evolving smooth metric tensor which satisfies

$$\partial_t \mathrm{g}(\dot\gamma(s),\dot\gamma(s))\big|_{t=0} = -2\mathrm{Ric}_{\mathrm{g}}(\dot\gamma(s),\dot\gamma(s)),$$

for almost-every $s \in [0,1]$ along g-geodesics γ. That is, g_t is *tangential* to the Ricci flow in this weak sense. This work of Gigli-Mantegazza gives one direction in which one could possibly define a Ricci flow for general metric measure spaces for all $t > 0$.

The latter correspondence is obtained by writing an evolving metric tensor via a partial differential equation. First, at each $t > 0$, $x \in \mathcal{M}$ and $v \in \mathrm{T}_x\mathcal{M}$, they consider the *continuity equation*

$$\text{(CE)} \qquad \begin{aligned} &-\mathrm{div}_{\mathrm{g}}(\rho^{\mathrm{g}}_t(x,y)\nabla\varphi_{t,x,v}(y)) = (\mathrm{d}_x\rho^{\mathrm{g}}_t(x,y))(v)\\ &\int_{\mathcal{M}} \varphi_{t,x,v}(y)\ d\mu_{\mathrm{g}}(y) = 0. \end{aligned}$$

For smooth metrics, the existence and regularity of solutions to this flow are immediate and they define a smooth family of metrics evolving in time by

$$\mathrm{g}_t(u,v)(x) = \int_{\mathcal{M}} \mathrm{g}(\nabla\varphi_{t,x,u}(y), \nabla\varphi_{t,x,v}(y))\ \rho_t^{\mathrm{g}}(x,y)\ d\mu_{\mathrm{g}}(y). \tag{GM}$$

While the formulation of this flow for $\mathrm{RCD}(K,N)$ spaces gives the existence of a time evolving distance metric for a possible singular space, it reveals little regularity information in positive time. On the other hand, the evolving metric tensor is only specified when both the initial metric and underlying manifold are smooth.

One of the motivating questions of the current paper is to better understand the behaviour of this flow on manifolds with geometric singularities and hence, the question of regularity will be a primary focus. Since there are few tools in the setting of $\mathrm{RCD}(K,N)$ spaces that are sufficiently mature to extract regularity information, we restrict ourselves exclusively to compact manifolds that are *smooth*, by which we assume only that they admit a smooth differential structure. While this may seem a potentially severe restriction, we vindicate ourselves by allowing for the metric tensor to be *rough*, i.e., a symmetric, positive-definite, $(2,0)$-tensor field with measurable coefficients. Rough metrics and their salient features are discussed in §2.1.

Such metrics allow a wide class of phenomena, so large that such a metric may not even induce a length structure, only an n-dimensional measure. Moreover, they may induce spaces that are not RCD. However, this potentially outrageous behaviour is redeemed by the fact that they are able to capture a wide class of *geometric singularities*, including Lipschitz transformations of C^1 geometries, conical singularities, and Euclidean boxes. These objects are considered in §3.2.

Our primary concern is when a metric exhibits singular behaviour on some closed subset $\mathcal{S} \neq \mathcal{M}$, but has good regularity properties on the open set $\mathcal{M} \setminus \mathcal{S}$. This is indeed the case for a Euclidean box, or a smooth compact manifold with finite number of geometric conical singularities. In this situation, away from the singular part, we are able to provide a metric tensor g_t. We say that two points $x, y \in \mathcal{M} \setminus \mathcal{S}$ are g_t-admissible if for any absolutely continuous curve $\gamma : I \to \mathcal{M}$ connecting these points, there is another absolutely continuous curve $\gamma' : I \to \mathcal{M}$ between x and y with length (measured via d_t) less than γ and for which $\gamma'(s) \in \mathcal{M} \setminus \mathcal{S}$ for almost-every s. For such a pair of points, we assert that the distance $\mathrm{d}_t(x,y)$, given by the $\mathrm{RCD}(K,N)$-flow of Gigli and Mantegazza, is induced by the metric tensor g_t. The following is a more precise showcasing of our main theorem. It is proved in §7.

Theorem 1.1. *Let $\mathcal{M}$ be a smooth, compact manifold with rough metric g that induces a distance metric d_{g}. Moreover, suppose there exists $K \in \mathbb{R}$ and $N > 0$ such that $(\mathcal{M}, \mathrm{d}_{\mathrm{g}}, \mu_{\mathrm{g}}) \in \mathrm{RCD}(K,N)$. If $\mathcal{S} \neq \mathcal{M}$ is a closed set and $\mathrm{g} \in \mathrm{C}^k(\mathcal{M} \setminus \mathcal{S})$, there exists a family of metrics $\mathrm{g}_t \in \mathrm{C}^{k-1,1}$ on $\mathcal{M} \setminus \mathcal{S}$ evolving according to (GM) on $\mathcal{M} \setminus \mathcal{S}$. For two points $x, y \in \mathcal{M}$ that are g_t-admissible, the distance $\mathrm{d}_t(x,y)$ given by the $\mathrm{RCD}(K,N)$ Gigli-Mantegazza flow is induced by g_t.*

The *divergence form* structure is a quintessential feature that allows for the analysis of the defining continuity equation (CE). In the compact case, it turns out that near every rough metric g, there is a smooth metric $\tilde{g}$ in a suitable L^∞-sense. Coupling this with the divergence structure, we are able to *perturb* this problem to the study of a divergence form operator with bounded, measurable coefficients on $\tilde{g}$ of the form

$$-\operatorname{div}_{\tilde{g}} \rho_t^{g}(x,)B\theta\nabla\varphi_{t,x,v} = \theta d_x(\rho_t^{g}(x,\cdot))(v),$$

where B is a bounded, measurable $(1,1)$-tensor transforming $\tilde{g}$ to g and θ the Raydon-Nikodym derivative of the two induced measures $\mu_{\tilde{g}}$ and μ_{g}.

This trick of "hiding" singularities in an operator has its origins to the investigations of boundary value problems with low regularity boundary. For us, this philosophy has a geometric reincarnation arising from investigations of the Kato square root problem on manifolds, with its origins in a seminal paper [4] by Axelsson, Keith and McIntosh and more recently by Morris in [22], and Bandara and McIntosh in [6].

In §4, we study the existence and regularity of solutions to such continuity equations by using spectral methods and PDE tools. We further emphasise that such equations allow for a certain *disintegration* - that is, at each point x, we solve a differential equation in the y variable. Eventually, we are concerned with objects involving an integration in y, and hence, we are able to allow weak solutions in y while being able to prove stronger regularity results in x.

While the reduction of a nonlinear problem to a pointwise linear one is a tremendous boon to the analysis that we conduct in this paper, there is a price to pay. The equation (CE) is nonlinear in x, and this nonlinear behaviour requires the analysis in x of the family of operators

$$x \mapsto \operatorname{div}_{\tilde{g}} \rho_t^{g}(x,\cdot)B\theta\nabla.$$

This is not as significant a disadvantage as one initially anticipates as this opens up the possibility to attacking regularity questions by the means of operator theory.

One of the main points of this paper is to illustrate how the regularity properties of the flow (GM) relate to the regularity properties of the heat kernel. Theorem 1.1 allows for the possibility of the evolving metric to become *less* regular than the original metric on the non-singular subset. An inspection of the continuity equation (CE) shows that the solution at a point x depends on sets of full measure potentially far away from this point. Thus, it is possible that singularities may resolve from smoothing properties emerging from the heat kernel. However, it is also possible that potentially unruly behaviour somewhere in heat kernel forces the flow to introduce additional singularities. That being said, we show that for $k \geq 1$, C^k metrics will continue to be C^k under the flow. We discuss these results and surrounding issues in greater depth in §3.

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2. Geometric singularities

Throughout this paper, by the term *geometric singularity*, we shall mean singularities that arise in the metric g of a *smooth manifold* $\mathcal{M}$. We allow such singularities to be a lack of differentiability or even the lack of continuity.

To emphasise this point, we contrast this to pure topological singularities, which are singularities that live in the topology and cannot be smoothed and transferred into the metric.

Throughout this paper, we let $\mathcal{M}$ to be a smooth manifold (possibly non-compact) of dimension $\dim \mathcal{M} = n$. By this, we mean a second countable, Hausdorff space that is locally Euclidean, with the transition maps being smooth.

On an open subset $\Omega \subset \mathcal{M}$, we write $\mathrm{C}^{k,\alpha}(\Omega)$ ($k \geq 0$ and $\alpha \in [0, 1]$) to mean k-times continuously differentiable functions bounded *locally* in coordinate patches inside Ω, and where the k-th partial derivatives are α-Hölder continuous *locally*. We write $\mathrm{C}^k(\Omega)$ instead of $\mathrm{C}^{k,0}(\Omega)$.

Let $\mathcal{T}^{(p,q)}\mathcal{M}$ denote the tensors of covariant rank p and contravariant rank q. We write $\mathrm{T}^*\mathcal{M} = \mathcal{T}^{(1,0)}\mathcal{M}$ and $\mathrm{T}\mathcal{M} = \mathcal{T}^{(0,1)}\mathcal{M}$, the *cotangent* and *tangent* bundles respectively. The bundle of differentiable k-forms are given by $\mathbf{\Omega}^k\mathcal{M}$ and the exterior algebra is given by $\mathbf{\Omega}\mathcal{M} = \oplus_{k=0}^n \mathbf{\Omega}^k\mathcal{M}$, where $\mathcal{M} \times \mathbb{R} = \mathbf{\Omega}^0\mathcal{M}$ (the bundle of functions), and $\mathrm{T}^*\mathcal{M} = \mathbf{\Omega}^1\mathcal{M}$.

The differentiable structure of the smooth manifold affords us with a differential operator $\mathrm{d} : \mathrm{C}^\infty(\mathbf{\Omega}^k\mathcal{M}) \to \mathrm{C}^\infty(\mathbf{\Omega}^{k+1}\mathcal{M})$. Indeed, this differential operator is dependent on the differentiable structure we associate to the manifold. We remark on this fact since, in dimensions higher than 4, there are homeomorphic differentiable structures that are not diffeomorphic (cf. [20] by Milnor and [10] by Freedman). From this point onward , we fix a differentiable structure on $\mathcal{M}$. We shall only exercise interest in the case of $k = 0$ where $\mathrm{d} : \mathrm{C}^\infty(\mathcal{M}) \to \mathrm{C}^\infty(\mathrm{T}^*\mathcal{M})$ and sometimes use the notation ∇ to denote d.

We emphasise that a smooth manifold also affords us with a measure structure. We say that a set $A \subset \mathcal{M}$ is measurable if for any chart (ψ, U) with $U \cap A \neq \varnothing$, we obtain that $\psi(U \cap A) \subset \mathbb{R}^n$ is Lebesgue measurable. By second countability, this quantification can be made countable. By writing $\mathbf{\Gamma}(\mathcal{T}^{(p,q)}\mathcal{M})$ we denote the sections of the vector bundle $\mathcal{T}^{(p,q)}\mathcal{M}$ with *measurable* coefficients.

2.1. **Rough metrics.**

In connection with investigating the geometric invariances of the Kato square root problem, Bandara introduced a notion of a *rough metric* in [5]. This notion is of fundamental importance to the rest of this paper and therefore, in this sub-section, we will describe some of the important features of such metrics. We do not assume that $\mathcal{M}$ is compact until later in this section.

Let us first recall the definition of a rough metric.

Definition 2.1. *Let* $\mathrm{g} \in \mathbf{\Gamma}(\mathcal{T}^{(2,0)}\mathcal{M})$ *be symmetric. Suppose that for each* $x \in \mathcal{M}$*, there exists some chart* (ψ, U) *and a constant* $C \geq 1$ *(dependent on* U*), such that, for* y*-a.e. in* U*,*

$$C^{-1}|u|_{\psi^*\delta(y)} \leq |u|_{\mathrm{g}(y)} \leq C|u|_{\psi^*\delta(y)},$$

where $u \in \mathrm{T}_y\mathcal{M}$*,* $|u|^2_{\mathrm{g}(y)} = \mathrm{g}(u,u)$ *and* $\psi^*\delta$ *is the pullback of the Euclidean metric inside* $\psi(U) \subset \mathbb{R}^n$*. Such a chart is said to satisfy the* local comparability condition.

It is easy to see that by taking U to be the pullback of a Euclidean ball contained in a chart near x, every $\mathrm{C}^{k,\alpha}$ metric (for $k \geq 0$ and $\alpha \in [0,1]$) is a rough metric.

Two rough metrics g and $\tilde{\mathrm{g}}$ are said to be C-close (for $C \geq 1$) if

$$C^{-1}|u|_{\mathrm{g}(x)} \leq |u|_{\tilde{\mathrm{g}}(x)} \leq C|u|_{\mathrm{g}(x)}$$

for almost-every $x \in \mathcal{M}$. If we assume that $\mathcal{M}$ is compact, then it is easy to see that for any rough metric g, there exists a constant $C \geq 1$ and a smooth metric $\tilde{\mathrm{g}}$ such that g and $\tilde{\mathrm{g}}$ are C-close. Two continuous metrics are C-close if the C-close condition above holds everywhere. Moreover, we note the following. Its proof is given in Proposition 3.14 in [5].

Proposition 2.2. *Let* g *and* $\tilde{\mathrm{g}}$ *be two rough metrics that are* C*-close. Then, there exists* $\mathrm{B} \in \mathbf{\Gamma}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$ *such that it is symmetric, almost-everywhere positive, invertible, and*

$$\tilde{\mathrm{g}}_x(\mathrm{B}(x)u, v) = \mathrm{g}_x(u,v)$$

for almost-every $x \in \mathcal{M}$*. Furthermore, for almost-every* $x \in \mathcal{M}$*,*

$$C^{-2}|u|_{\tilde{\mathrm{g}}(x)} \leq |\mathrm{B}(x)u|_{\tilde{\mathrm{g}}(x)} \leq C^2|u|_{\tilde{\mathrm{g}}(x)},$$

and the same inequality holds with $\tilde{\mathrm{g}}$ *and* g *interchanged. If* $\tilde{\mathrm{g}} \in \mathrm{C}^k$ *and* $\mathrm{g} \in \mathrm{C}^l$ *(with* $k, l \geq 0$*), then the properties of* B *are valid for all* $x \in \mathcal{M}$ *and* $\mathrm{B} \in \mathrm{C}^{\min\{k,l\}}(\mathrm{T}^*\mathcal{M} \otimes \mathrm{T}\mathcal{M})$.

A rough metric always induces a measure described by the expression

$$d\mu_{\mathrm{g}}(x) = \sqrt{\det(\mathrm{g}^{ij}(x))}\ d\mathscr{L}(x)$$

inside a locally comparable chart. The well-definedness of this expression is verified just as in the case of a $\mathrm{C}^{k,\alpha}$ metric. This measure can easily be proven to be Borel and finite on compact sets. The notion of measurable which we have defined agrees with the notion of μ_{g}-measurable obtained via a rough metric. Moreover, the following holds for two C-close metrics.

Proposition 2.3. *Let* g *and* $\tilde{\mathrm{g}}$ *be* C*-close for some* $C \geq 1$*. Then, the measure* $d\mu_{\mathrm{g}}(x) = \sqrt{\det \mathrm{B}(x)}\ d\mu_{\tilde{\mathrm{g}}}(x)$ *for* x*-a.e., and* $C^{-\frac{n}{2}}\mu_{\mathrm{g}} \leq \mu_{\tilde{\mathrm{g}}} \leq C^{\frac{n}{2}}\mu_{\mathrm{g}}$*. Moreover, if* $\tilde{\mathrm{g}}$ *is continuous, then the measure* μ_{g} *is Radon.*

Proof. The first part of the statement is proved as Proposition 3.16 [5]. To prove that μ_{g} is Radon, so that $\kappa\mu_{\tilde{\mathrm{g}}} \leq \mu_{\mathrm{g}} \leq \mu_{\tilde{\mathrm{g}}}$. Now, since $\mu_{\tilde{\mathrm{g}}}$ is inner-regular, for a Borel $B \subset \mathcal{M}$ and every $\varepsilon > 0$ there exists $K_\varepsilon \subset B$ such that $\mu_{\tilde{\mathrm{g}}}(B) - \mu_{\tilde{\mathrm{g}}}(K_\varepsilon) \leq \varepsilon$. Now, note that

$$\mu_{\mathrm{g}}(B) - \mu_{\mathrm{g}}(K_\varepsilon) \leq C^{\frac{n}{2}}\mu_{\tilde{\mathrm{g}}}(B) - C^{-\frac{n}{2}}\mu_{\tilde{\mathrm{g}}}(K_\varepsilon) \leq C'\mu_{\tilde{\mathrm{g}}}(B) - \mu_{\tilde{\mathrm{g}}}(K_\varepsilon) \leq C'\varepsilon,$$

where $C' = \max\left\{C^{\frac{n}{2}}, C^{-\frac{n}{2}}\right\}$. Thus, $\mu_{\mathrm{g}}(B) = \sup_{K \Subset B} \mu_{\mathrm{g}}(K)$. □

We remark that throughout the paper, when we say that $(\mathcal{M}, \mathrm{g})$ *induces a length structure*, we mean that between any two points $x, y \in \mathcal{M}$ there exists a differentiable curve $\gamma : I \to \mathcal{M}$ with $\gamma(0) = x$, $\gamma(1) = y$ such that

$$0 < \int_I |\dot{\gamma}(t)|_{\mathrm{g}(\gamma(t))} < \infty.$$

Then, the induced distance $\mathrm{d}_{\mathrm{g}}(x, y)$ is simply given as in the smooth case by taking an infimum over all curves between such points.

2.2. L^∞**-metrics and metrics of divergence form operators.** The goal of this subsection is to illustrate the connections of rough metrics to other low-regularity metrics that are often mentioned in the folklore. In fact, we shall see that as a virtue of compactness, these notions are indeed equivalent. This section is intended as motivation for us considering rough metrics and can be safely omitted.

First, we highlight the following simple lemma.

Lemma 2.4. *Suppose that* $\mathrm{g} \in \mathbf{\Gamma}(\mathcal{T}^{(2,0)}\mathcal{M})$ *is symmetric and that there exists a smooth metric* h *and* $C \geq 1$ *such that*

$$C^{-1}|u|_{\mathrm{h}(x)} \leq |u|_{\mathrm{g}(x)} \leq C|u|_{\mathrm{h}(x)}$$

for almost-every $x \in \mathcal{M}$*. Then* g *is a rough metric.*

Proof. Fix $x \in \mathcal{M}$ and let (ψ, U) be a chart near $x \in \mathcal{M}$. Let $V = \psi^{-1}(B_r(\psi(x)))$, where $B_r(\psi(x)) \subset \mathbb{R}^n$ is a Euclidean ball with $r > 0$ chosen such that $\overline{B_r(\psi(x))} \subset \psi(U)$. Then, by virtue of the smoothness of h and since $\overline{V}$ is compact, we obtain some $C_V \geq 1$ such that

$$C_V^{-1}|u|_{\psi^*\delta(y)} \leq |u|_{\mathrm{h}(y)} \leq C_V|u|_{\psi^*\delta(y)},$$

for all $y \in V$. On combining this with our hypothesis, we find that for almost-every $y \in V$,

$$(C_V C)^{-1}|u|_{\mathrm{h}(y)} \leq |u|_{\mathrm{g}(y)} \leq C_V C|u|_{\mathrm{h}(y)}.$$

That is, g is a rough metric. □

Next, we define the following notion of an L^∞-metric.

Definition 2.5. *We say that a symmetric* $\mathrm{g} \in \mathbf{\Gamma}(\mathcal{T}^{(2,0)}\mathcal{M})$ *is an* L^∞*-metric on* $\mathcal{M}$ *if:*

(i) there exists a $\mathrm{g}^{-1} \in \mathbf{\Gamma}(\mathcal{T}^{(0,2)}\mathcal{M})$ *inverse to* g*, by which we mean that writing* $G = (\mathrm{g}^{ij}(x))$ *and* $G' = (\mathrm{g}^{-1}_{ij}(x))$*,* $GG' = G'G = \mathrm{I}$ *for almost-every* $x \in \mathcal{M}$*, and*
(ii) there exists a smooth metric h *and constants* $\Lambda_1, \Lambda_2 > 0$ *such that* $|\mathrm{g}|_{\mathrm{h}} \leq \Lambda_1$ *and* $|\mathrm{g}^{-1}|_{\mathrm{h}} \leq \Lambda_2$*.*

We prove that an L^∞-metric is indeed a rough metric.

Proposition 2.6. *An* L^∞*-metric* g *is a rough metric. It is* $(\max\{\Lambda_1 n, \Lambda_2 n\})$*-close to a smooth metric* h*.*

Proof. Fix $x \in \mathcal{M}$ in which the inequalities in the definition of an L^∞ metric is valid. Let $\{e^i\}$ be a frame for $\mathrm{T}_x\mathcal{M}$ so that $\mathrm{h}_{ij}(x) = \delta_{ij}$. Let $G = (\mathrm{g}^{ij}(x))$ Then, note that

$$\Lambda_1 \geq |\mathrm{g}(x)|^2_{\mathrm{h}(x)} = \mathrm{g}^{ij}(x)\mathrm{g}^{mn}(x)\mathrm{h}_{im}(x)\mathrm{h}_{jn} = \sum_{ij} |\mathrm{g}^{ij}|^2.$$

Now, let $u \in \mathrm{T}_x\mathcal{M}$, and then

$$\begin{aligned}|u|^2_{\mathrm{g}(x)} = \mathrm{g}(x)(u,u) = \mathrm{g}^{ij}u_iu_j \leq \sum_j |\sum_i \mathrm{g}^{ij}u_iu_j| \\ \leq \sum_j \left(\sum_i |\mathrm{g}^{ij}u_j|^2\right)^{\frac{1}{2}} \left(\sum_i |u_i|^2\right)^{\frac{1}{2}},\end{aligned}$$

where the last inequality follows form the Cauchy Schwarz inequality. Now, by our previous calculation, we have that $|\mathrm{g}^{ij}|^2 \leq \Lambda_1$, and hence,

$$\sum_j \sum_i |\mathrm{g}^{ij}u_j|^2 \leq \Lambda_1 n \sum_j |u_j|^2 = \Lambda_1 n |u|^2_{\mathrm{h}(x)}.$$

That is, $|u|_{\mathrm{g}(x)} \leq \sqrt{\Lambda_1 n}|u|_{\mathrm{h}(x)}$. Applying this with g^{-1} in place of g and h^{-1} in place of h, we further obtain that $|u|_{\mathrm{g}^{-1}(x)} \leq \sqrt{\Lambda_1 n}|u|_{\mathrm{h}^{-1}(x)}$.

Now, we note that $\tilde{G} = (\mathrm{g}^{-1}_{ij}(x)) = G^{-1}$. Since G is symmetric, let $G = PDP^{\mathrm{tr}}$, its eigenvalue decomposition. Indeed, $D = \mathrm{diag}(\lambda_i)$ and $\lambda_i > 0$ since G is invertible. On letting $D^{-1} = \mathrm{diag}(\sigma_i)$, note that $\sigma_n = \lambda_1$ and we have that $\sigma_n \leq \Lambda_2 n$, which means that $\lambda_1 \leq \frac{1}{\Lambda_2 n}$. So, now

$$\begin{aligned}|u|^2_{\mathrm{g}(x)} &= u^{\mathrm{tr}}Gu = u^{\mathrm{tr}}PDP^{\mathrm{tr}}u = (P^{\mathrm{tr}}u)^{\mathrm{tr}}D(P^{\mathrm{tr}}u) = \tilde{u}^{\mathrm{tr}}D\tilde{u} \\ &= \sum_i \lambda_i|\tilde{u}_i| \geq \min\{\lambda_i\}\sum_i |\tilde{u}_i| \geq \frac{1}{\Lambda_2 n}|P^{\mathrm{tr}}u| = \frac{1}{\Lambda_2 n}|u|^2,\end{aligned}$$

since P is an orthonormal matrix. That is, we have shown that for almost-every $x \in \mathcal{M}$, and every $u \in \mathrm{T}_x\mathcal{M}$,

$$(\Lambda_2 n)^{-1}|u|_{\mathrm{g}(x)} \leq |u|_{\mathrm{h}(x)} \leq (\Lambda_1 n)^{-1}|u|_{\mathrm{h}(x)}.$$

Thus, by Lemma 2.4, g is a rough metric. □

Another class of metrics we consider are metrics arising from coefficients of elliptic operators in divergence form. In particular, see the paper [25] by Saloff-Coste, where the author explicitly considers this class of metrics, although he makes a qualitative assumption that the coefficients are smooth.

Fix some smooth metric h and let $A \in \mathbf{\Gamma}(\mathcal{T}^{(1,1)}\mathcal{M}, \mathrm{h})$ be symmetric. Consider the real symmetric form $J_A[u,v] = \langle A\nabla u, \nabla v\rangle_{\mathrm{h}}$. In order for this to define an elliptic operator, the natural assumption is to ask that there exist $\kappa_1, \kappa_2 > 0$ such that

$$\mathrm{h}(Au,u)(x) \geq \kappa_1 |u|^2_{\mathrm{h}(x)} \quad \text{and} \quad |A|_{\mathrm{h}(x)} \leq \kappa_2,$$

for almost-every $x \in \mathcal{M}$. For the sake of nomenclature, let us say that the coefficients A are *elliptic* if this condition is met. Under these conditions, there exists a self-adjoint operator associated to J_A which is $\mathrm{L}_A u = \mathrm{div}_{\mathrm{h}} A\nabla u$. Such operators have been amply studied in the literature.

Let us now define a metric associated to elliptic coefficients A by writing $\mathrm{g}(u,v) = \mathrm{h}(Au,v)$. Then, we have the following proposition.

Proposition 2.7. *A metric* g *induced from elliptic coefficients* A *via a smooth metric* h *is a rough metric. The metric* g *is* $(\max\{\kappa_1,\kappa_2\})$*-close to* h*.*

Proof. By virtue of the fact that A are elliptic coefficients, we immediately obtain that for almost-every $x \in \mathcal{M}$, $|u|^2_{\mathrm{g}(x)} = \mathrm{h}(Au,u) \geq \kappa_1^2 |u|^2_{\mathrm{h}(x)}$ for every $u \in \mathrm{T}_x\mathcal{M}$.

For the upper bound, fix an x where the ellipticity inequality is valid, and choose a frame so that let $\mathrm{h}_{ij}(x) = \delta_{ij}$. Then, we have that $|A|^2_{\mathrm{h}(x)} = \sum_{ij} |A^i_j|^2$. Then,

$$|u|^2_{\mathrm{g}(x)} = \mathrm{h}_x(Au,u) |Au|_{\mathrm{h}(x)} |u|_{\mathrm{h}(x)}.$$

Now,

$$|Au|^2_{\mathrm{h}(x)} = \sum_j |\sum_i A^i_j u_i|^2 \leq \sum_j \left(\sum_i |A^i_j|^2\right)\left(\sum_i |u_i|^2\right) \leq \kappa_2^4 |u|^2_{\mathrm{h}(x)}.$$

Thus, $|u|^2_{\mathrm{g}(x)} \leq \kappa_2^2 |u|^2_{\mathrm{h}(x)}$. By invoking Lemma 2.4, we obtain that g is a rough metric. □

As a finale, on collating our results here, we present the following proposition.

Proposition 2.8. *We have the following:*

(i) every rough metric that is close to a smooth one is an L^∞ *metric,*
(ii) every L^∞ *metric is a metric induced via elliptic coefficients,*
(iii) every metric induced via elliptic coefficients is an L^∞ *metric.*

If $\mathcal{M}$ *is compact, then all these notions are equivalent.*

Proof. For (i), suppose that g is a rough metric and that it is C-close to a smooth metric h. Then, by Proposition 2.2, we obtain a $B \in \mathbf{\Gamma}(\mathcal{T}^{(1,1)}\mathcal{M})$ so that $\mathrm{g}(u,v) = \mathrm{h}(Bu,v)$ and

$$C^{-2}|u|_{\mathrm{h}(x)} \leq |B(x)u|_{\mathrm{h}(x)} \leq C^2 |u|_{\mathrm{h}(x)},$$

for almost-every x. Fix an x where this inequality is valid and choose a h orthonormal frame $\{e_i\}$ at x. Then,

$$|\mathrm{g}(x)|^2_{\mathrm{h}(x)} = \sum_{ij} |\mathrm{g}^{ij}(x)|^2 = \sum_{ij} |\mathrm{h}(Be_i, e_j)| \leq \sum_{ij} |Be_i|_{\mathrm{h}(x)} |e_j|_{\mathrm{h}(x)} \leq n^2 C^2.$$

So, $|\mathrm{g}|_{\mathrm{h}} \leq n^2 C^2$ almost-everywhere. Since $\mathrm{g}^{-1}(u,v) = \mathrm{h}^{-1}(B^{-1}u, v)$, a similar calculation shows that $|\mathrm{g}^{-1}|_{\mathrm{h}^{-1}} \leq n^2 C^2$.

To prove (ii), suppose that g is an L^∞ metric. Then, we have shown in Proposition 2.6 that it is a rough metric that is close to a smooth metric h. Hence, by Proposition 2.2, we have $B \in \mathbf{\Gamma}(\mathcal{T}^{(1,1)}\mathcal{M})$ which can easily be checked to satisfy ellipticity. Hence, $\mathrm{g}(u,v) = \mathrm{h}(Bu, v)$, i.e., it is a metric induced by elliptic coefficients. Then, it is a rough metric that is C-close to a smooth one and by (i), we obtain that it is an L^∞ metric.

If further we assume that $\mathcal{M}$ is a compact manifold, then near every rough metric g, there is a smooth metric h, and hence, by (i), we obtain that every rough metric is L^∞ or equivalently, defined via elliptic coefficients. □

In particular, this proposition gives credence to the notion of a rough metric since it is a sufficiently general notion that is able to capture the behaviour of these other aforementioned low regularity metrics.

2.3. **Lebesgue and Sobolev space theory.** A more pertinent feature of rough metrics is that they admit a Sobolev space theory. In order to make our exposition shorter and more accessible, from here on, we assume that $\mathcal{M}$ is *compact*. First, we note that since $(\mathcal{M}, \mu_{\mathrm{g}})$ is a measure space, we obtain a Lebesgue theory. Let $\mathrm{L}^p(\mathcal{T}^{(p,q)}\mathcal{M}, \mathrm{g})$ denote the p-integrable Lebesgue spaces over the bundle of (p,q) tensors. We write $\mathrm{L}^p(\mathcal{M}, \mathrm{g})$ for the case that $p = q = 0$. We quote the following result which is listed as Proposition 3.10 in [5].

Proposition 2.9. *For a rough metric* g*,* $\nabla_p : \mathrm{C}^\infty \cap \mathrm{L}^p(\mathcal{M}) \to \mathrm{C}^\infty \cap \mathrm{L}^p(\mathrm{T}^*\mathcal{M})$ *and* $\nabla_c : \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M}) \to \mathrm{C}^\infty_{\mathrm{c}}(\mathrm{T}^*\mathcal{M})$ *given by* $\nabla_p = \mathrm{d}$ *and* $\nabla_c = \mathrm{d}$ *on the respective domains are closable, densely-defined operators.*

As a consequence, we define the Sobolev spaces as function spaces by writing $\mathrm{W}^{1,p}(\mathcal{M}) = \mathcal{D}(\overline{\nabla_p})$ and $\mathrm{W}^{1,p}_0 = \mathcal{D}(\overline{\nabla_c})$.

Proposition 2.10. *Let* g *and* $\tilde{\mathrm{g}}$ *be two* C*-close rough metrics on a compact manifold* $\mathcal{M}$*. Then,*

(i) whenever $p \in [1,\infty)$*,* $\mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g}) = \mathrm{L}^p(\mathcal{T}^{(r,s)}\mathcal{M}, \tilde{\mathrm{g}})$ *with*

$$C^{-\left(r+s+\frac{n}{2p}\right)} \|u\|_{p,\tilde{\mathrm{g}}} \leq \|u\|_{p,\mathrm{g}} \leq C^{r+s+\frac{n}{2p}} \|u\|_{p,\tilde{\mathrm{g}}},$$

(ii) for $p = \infty$*,* $\mathrm{L}^\infty(\mathcal{T}^{(r,s)}\mathcal{M}, \mathrm{g}) = \mathrm{L}^\infty(\mathcal{T}^{(r,s)}\mathcal{M}, \tilde{\mathrm{g}})$ *with*

$$C^{-(r+s)} \|u\|_{\infty,\tilde{\mathrm{g}}} \leq \|u\|_{\infty,\mathrm{g}} \leq C^{r+s} \|u\|_{\infty,\tilde{\mathrm{g}}},$$

(iii) *the Sobolev spaces* $\mathrm{W}^{1,p}(\mathcal{M},\mathrm{g}) = \mathrm{W}^{1,p}(\mathcal{M},\tilde{\mathrm{g}}) = \mathrm{W}^{1,p}_0(\mathcal{M},\mathrm{g}) = \mathrm{W}^{1,p}_0(\mathcal{M},\tilde{\mathrm{g}})$ *with*

$$C^{-\left(1+\frac{n}{2p}\right)}\|u\|_{\mathrm{W}^{1,p},\tilde{\mathrm{g}}} \le \|u\|_{\mathrm{W}^{1,p},\mathrm{g}} \le C^{1+\frac{n}{2p}}\|u\|_{\mathrm{W}^{1,p},\tilde{\mathrm{g}}},$$

(iv) *the Sobolev spaces* $\mathrm{W}^{\mathrm{d},p}(\mathcal{M},\mathrm{g}) = \mathrm{W}^{\mathrm{d},p}(\mathcal{M},\tilde{\mathrm{g}})$ *with*

$$C^{-\left(n+\frac{n}{2p}\right)}\|u\|_{\mathrm{W}^{\mathrm{d},p},\tilde{\mathrm{g}}} \le \|u\|_{\mathrm{W}^{\mathrm{d},p},\mathrm{g}} \le C^{n+\frac{n}{2p}}\|u\|_{\mathrm{W}^{\mathrm{d},p},\tilde{\mathrm{g}}},$$

(v) *the divergence operators satisfy* $\mathrm{div}_{\mathrm{g}} = \theta^{-1}\,\mathrm{div}_{\tilde{\mathrm{g}}}\,\theta\mathrm{B}$.
(vi) *the Laplacians satisfy* $\Delta_{\mathrm{g}} = -\theta^{-1}\,\mathrm{div}_{\tilde{\mathrm{g}}}\,\theta\mathrm{B}\nabla$.

We emphasise (iii), which demonstrates that $\mathrm{W}^{1,2}(\mathcal{M},\mathrm{g}) = \mathrm{W}^{1,2}(\mathcal{M},\tilde{\mathrm{g}})$ for any rough metric g, since, as we have aforementioned, compactness guarantees the existence of a smooth metric $\tilde{\mathrm{g}}$ that is C-close to g.

3. Main results and applications

3.1. **Existence and regularity of the flow.** The broader perspective underpinning our analysis in this paper is to relate the regularity of the heat kernel to the Gigli-Mantegazza flow. Indeed, this is to be expected simply from inspection of the main governing equation (CE) for this flow.

In §4, we consider L^∞-coefficient differential operators on smooth manifolds, and we obtain solutions to more general equations similar to (CE). Furthermore, we conduct operator theory of operators of the type $x \mapsto \mathrm{div}_{\mathrm{g}}\,\omega_x\nabla$ in order to define a notion of derivative that is weak enough to account for the lack of regularity of the coefficients ω_x but sufficiently strong enough to be useful to demonstrate the regularity of the the flow (GM). In §5, we prove some auxiliary facts needed to ensure that (GM) indeed does define a Riemannian metric, and on coupling our main results from §4, we obtain the following theorem. It is the most general geometric result that we showcase in this paper. Its proof can be found in §5.2.

Theorem 3.1. *Let* $\mathcal{M}$ *be a smooth, compact manifold and* g *a rough metric. Suppose that the heat kernel* $(x,y) \mapsto \rho^{\mathrm{g}}_t(x,y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$ *and that on an open set* $\varnothing \neq \mathcal{N}$, $(x,y) \mapsto \rho^{\mathrm{g}}_t(x,y) \in \mathrm{C}^k(\mathcal{N}^2)$ *where* $k \geq 2$. *Then, for* $t > 0$, g_t *is a Riemannian metric on* $\mathcal{N}$ *of regularity* $\mathrm{C}^{k-2,1}$.

We remark that allowing for a Lipschitz heat kernel is neither a restriction nor is it too general. We will see in the following section that the most important class of objects we consider, namely when $(\mathcal{M},\mathrm{g})$ is an $\mathrm{RCD}(K,N)$ space, will admit such a heat kernel.

The reader may find it curious that, even though we assume that the heat kernel is C^k away from the singular region, and only a single derivative of the heat kernel appears in the source term of (CE), we are only able to assert that the resulting flow is $\mathrm{C}^{k-2,1}$. In a sense, it is because the global regularity of the heat kernel, which is only Lipschitz, becomes significant in proving the continuity of the $(k-1)$ partial derivatives.

We remark that it may be possible to assert this continuity performing the operator theory of $x \mapsto \mathrm{div}_{\mathrm{g}}\, \rho^{\mathrm{g}}_t(x,\cdot)\nabla$ with greater care than we have done.

For a C^1 global heat kernel, we are able to assert that the $(k-1)$ partial derivatives are indeed continuous. This is the content of the following theorem. Note that, unlike Theorem 3.1 where we assumed that the heat kernel was at least twice continuously differentiable on the non-singular part, we allow for heat kernels with only a single derivative on the non-singular region. This theorem is an immediate consequence of Theorem 6.3 in §6.1.

Theorem 3.2. *Let $\mathcal{M}$ be a smooth, compact manifold and* g *a rough metric. Suppose that the heat kernel $(x,y) \mapsto \rho^{\mathrm{g}}_t(x,y) \in \mathrm{C}^1(\mathcal{M}^2)$ and that on an open set $\varnothing \neq \mathcal{N}$, $(x,y) \mapsto \rho^{\mathrm{g}}_t(x,y) \in \mathrm{C}^k(\mathcal{N}^2)$ where $k \geq 1$. Then, for $t > 0$, g_t is a Riemannian metric on $\mathcal{N}$ of regularity C^{k-1}.*

Typically, in an open region where the metric is C^k for $k \geq 1$, we expect the heat kernel to improve to C^{k+1}. This is an immediate consequence of the fact that the region is open, and because we can write the Laplacian via a change of coordinates as a non-divergence form equation with C^{k-1} coefficients. We then obtain regularity via Schauder theory. This analysis is conducted in §6.2. By considering the situation where a rough metric improves to a C^k metric away from a closed singular set, Theorem (3.1) yields a metric tensor away from the singular set that is of regularity $\mathrm{C}^{k-1,1}$. That is, the resulting flow may be *more* singular than the initial metric inside such a region. However, by coupling the results of §6.2 with Theorem 3.2, we are able to assert that the flow remains at least as regular as the initial metric.

Theorem 3.3. *Let $\mathcal{M}$ be a smooth, compact manifold and $\mathrm{g} \in \mathrm{C}^k$ for $k \geq 1$. Then, the flow $\mathrm{g}_t \in \mathrm{C}^k$ for each $t > 0$.*

We emphasise that we are not providing sharp regularity information via these theorems. That is, we are unable to assert that if the initial metric is $\mathrm{C}^k \setminus \mathrm{C}^{k-1}$, then the resulting flow is also $\mathrm{C}^k \setminus \mathrm{C}^{k-1}$. In fact, this may not be the case, it may be possible that in some instances, the flow g_t improves in regularity. These questions are open and beyond the scope of this paper.

3.2. **Applications to geometrically singular spaces.**

In this subsection, we consider geometric applications of Theorem 3.1, particular to spaces with geometric singularities.

As a start, we describe our notion of a *geometric conical singularity*. For that, let us first describe the n-cone of radius r and height h by

$$\mathcal{C}(r,h) = \left\{ (x,t) \in \mathbb{R}^{n+1} : |x| = \frac{r}{h}(h-t) : t \in [0,h] \right\}.$$

With this notation in hand, we define the following.

Definition 3.4 (Geometric conical singularities)**.** *Let $\mathcal{M}$ be a smooth manifold and* g *a rough metric. Let $\{p_1,\dots,p_k\} \subset \mathcal{M}$ and suppose there exists a charts (ψ_i, U_i) mutually disjoint such that $\mathrm{g} \in \mathrm{C}^k(\mathcal{M} \setminus \cup_i \overline{U}_i)$. Moreover, suppose for each i, there* *is a Lipeomorphism $F_i : U_i \to \mathcal{C}(r_i, h_i) \subset \mathbb{R}^{n+1}$ which improves to a C^{k+1} diffeomorphism (for $k \geq 1$) on $U_i \setminus \{p_i\}$ and that $\mathrm{g} = F_i^* \langle \cdot , \cdot \rangle_{\mathbb{R}^{n+1}}$ inside U_i. Then, we say that $(\mathcal{M}, \mathrm{g})$ is C^k-geometry with* geometric conical singularities *at points $\{p_1, \dots, p_k\}$.*

A direct consequence of Theorem 1.1 is then the following.

Corollary 3.5. *Let $(\mathcal{M}, \mathrm{g})$ be a C^k-geometry for $k \geq 1$ with geometric conical singularities at $\{p_1, \dots, p_k\}$. Then, the flow $\mathrm{g}_t \in \mathrm{C}^{k-1,1}$ and the induced metric coincides everywhere with the flow of the metrics d_t for $\mathrm{RCD}(K, N)$ spaces defined by Gigli-Mantegazza.*

Moreover, we are able to flow the sphere with a conical pole. See §7.2 for the construction and proof.

Corollary 3.6 (Witch's hat sphere)**.** *Let $(\mathrm{S}^n, \mathrm{g}_{\mathrm{witch}})$ be the sphere with a cone attached at the north pole. Then, the flow $\mathrm{g}_t \in \mathrm{C}^\infty$ and the induced metric agrees with the flow of the metrics d_t for $\mathrm{RCD}(K, N)$ spaces defined by Gigli-Mantegazza.*

Also, as a consequence of Theorem 1.1, we are able to consider the n-dimensional box in Euclidean space. Again, its proof is contained in §7.2.

Corollary 3.7. *Let (B, g) be an n-box. Then, the flow $\mathrm{g}_t \in \mathrm{C}^\infty$ away from $\mathcal{S}$, the set of edges and corners, induces the same distance as d_t, the $\mathrm{RCD}(K, N)$ Gigli-Mantegazza flow, for g_t-admissible points.*

We remark that a shortcoming of the flow in this situation is that we have not classified the g_t-admissible points. We do not expect this analysis to be straightforward since it involves understanding whether the flat pieces are preserved in some way under d_t. However, since we have supplied a metric g_t away from a set of measure zero, we expect it to be possible to induce d_t via this metric on a very large part of the box.

4. Elliptic problems and regularity

Throughout, let us fix $\varnothing \neq \mathcal{N} \subset \mathcal{M}$ to be an open subset of $\mathcal{M}$. To study the flow of Gigli-Mantegazza, we study a slightly more general elliptic problem than (CE). Let $\omega \in \mathrm{C}^{0,1}(\mathcal{M}^2)$ be a function satisfying $\omega(x, y) > 0$ for all $x, y \in \mathcal{M}$. Moreover, let $x \mapsto \omega(x, \cdot) \in \mathrm{C}^k(\mathcal{N})$ where $k \geq 1$. For convenience, we write $\omega(x, \cdot) = \omega_x$. Then, for $\eta \in \mathrm{L}^2(\mathcal{M})$ we want to solve for $\varphi \in \mathrm{W}^{1,2}(\mathcal{M})$ satisfying the equation

$$-\operatorname{div}_{\mathrm{g}} \omega_x \nabla \varphi = \eta. \tag{F}$$

In this section, we establish existence, uniqueness and regularity (in x) for this equation.

4.1. **L^∞-coefficient divergence form operators on smooth metrics.** Due to the lack of regularity of an arbitrary rough metric, we are forced to solve the the problem (F) via perturbation at the level of the metric. Indeed, this is not a small perturbation result, because we are not guaranteed the existence of arbitrarily close smooth metrics to a rough metric. Instead, we will have to contend ourselves to studying elliptic PDE of the form $-\operatorname{div} A\nabla$, where A are only bounded measurable coefficients defining an elliptic problem.

More precisely, throughout this subsection, we fix $\mathcal{M}$ to be a smooth compact manifold and $\tilde{\mathrm{g}}$ to be a smooth metric. Let $A \in \mathrm{L}^\infty(\mathcal{T}^{(1,1)}\mathcal{M}) = \mathrm{L}^\infty(\mathcal{L}(\mathcal{T}^{(1,0)}\mathcal{M}))$ be symmetric. That is, for almost-every x, in coordinates, $A(x)$ can be written as a symmetric matrix. Further, we assume that there exists $\kappa > 0$ such that $\langle Au, u\rangle \geq \kappa \|u\|^2$. That is, A is bounded below. Moreover, we quantify the L^∞ bound for A via assuming there exists $\Lambda > 0$ satisfying $\langle Au, u\rangle = \|\sqrt{A}\|^2 \leq \Lambda$

Let $J_A : \mathrm{W}^{1,2}(\mathcal{M}) \times \mathrm{W}^{1,2}(\mathcal{M}) \to \mathbb{R}_+$ be given by

$$J_A[u, v] = \langle A\nabla u, \nabla v\rangle = \int_{\mathcal{M}} \tilde{\mathrm{g}}_x(A(x)\nabla u(x), \nabla v(x))\ d\mu_{\tilde{\mathrm{g}}}(x).$$

Then, by the lower bound on A, we obtain that $J_A[u, u] \geq \kappa\|\nabla u\|^2$. The Lax-Milgram theorem then yields that there exists a unique, closed, densely-defined self-adjoint operator L_A with domain $\mathcal{D}(\mathrm{L}_A) \subset \mathrm{W}^{1,2}(\mathcal{M})$ such that $J_A[u, v] = \langle \mathrm{L}_A u, v\rangle$ for $u \in \mathcal{D}(\mathrm{L}_A)$ and $v \in \mathrm{W}^{1,2}(\mathcal{M})$. Moreover, since the form J_A is real-symmetric due to the symmetry of A, this theorem further yields that $\mathcal{D}(\sqrt{\mathrm{L}_A}) = \mathrm{W}^{1,2}(\mathcal{M})$ and $J_A[u, v] = \langle \sqrt{\mathrm{L}_A}u, \sqrt{\mathrm{L}_A}v\rangle$. The uniqueness then allows us to assert that $\mathrm{L}_A = -\operatorname{div}_{\tilde{\mathrm{g}}} A\nabla$.

For the remainder of this section, we rely on some facts from the spectral theory of sectorial and, more particularly, self-adjoint operators. We refer the reader to [16] by Kato, [1] by Albrecht, Duong and McIntosh, [15] by Gilbarg and Trudinger and [9] by Cowling, Doust, McIntosh and Yagi for a more detailed exposition on the connection between PDE and spectral theory.

As a first, we establish a spectral splitting for L_A. We recall that for $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathcal{M})$ $u_B = \fint_B u\ d\mu_{\tilde{\mathrm{g}}}$ for $B \subset \mathcal{M}$ a Borel set.

Proposition 4.1. *The space* $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\mathrm{L}_A) \oplus^\perp \overline{\mathcal{R}(\mathrm{L}_A)}$ *and the operator* L_A *restricted to either* $\mathcal{N}(\mathrm{L}_A)$ *or* $\overline{\mathcal{R}(\mathrm{L}_A)}$ *preserves each space respectively. Moreover,* $\mathcal{N}(\mathrm{L}_A) = \mathcal{N}(\nabla)$ *and* $\overline{\mathcal{R}(\mathrm{L}_A)} = \left\{u \in \mathrm{W}^{1,2}(\mathcal{M}) : \int_M u\ d\mu_{\tilde{\mathrm{g}}} = 0\right\}$.

Proof. The fact that the operator splits the space orthogonally to $\mathcal{N}(\mathrm{L}_A)$ and $\overline{\mathcal{R}(\mathrm{L}_A)}$, and that its restriction to each of these spaces preserves the respective space is a direct consequence of the fact that L_A is self-adjoint. See Theorem 3.8 in [9].

First, let us apply this same argument to the operator $\sqrt{\mathrm{L}_A}$, so that we obtain the splitting $\mathrm{L}^2(\mathcal{M}) = \mathcal{N}(\sqrt{\mathrm{L}_A}) \oplus^\perp \overline{\mathcal{R}(\sqrt{\mathrm{L}_A})}$. Now, fix $u \in \mathcal{N}(\sqrt{\mathrm{L}_A})$. Then, $\sqrt{\mathrm{L}_A}u = 0$ which implies that $u \in \mathcal{D}(\mathrm{L}_A)$ and $\mathrm{L}_A u = \sqrt{\mathrm{L}_A}(\sqrt{\mathrm{L}_A}u) = 0$. Hence, $u \in \mathcal{N}(\mathrm{L}_A)$.

For the reverse inclusion, suppose that $0 \neq u \in \mathcal{N}(\mathrm{L}_A)$. Then, $\sqrt{\mathrm{L}_A}\sqrt{\mathrm{L}_A}u = 0$. That is, $\sqrt{\mathrm{L}_A}u \in \mathcal{N}(\sqrt{\mathrm{L}_A})$. But since $\sqrt{\mathrm{L}_A}$ preserves $\mathcal{N}(\sqrt{\mathrm{L}_A})$, $u \in \mathcal{N}(\sqrt{\mathrm{L}_A})$. Thus, $\mathcal{N}(\mathrm{L}_A) = \mathcal{N}(\sqrt{\mathrm{L}_A})$. Now, since we have that for all $u \in \mathrm{W}^{1,2}(\mathcal{M})$, $\kappa\|\nabla u\|^2 \leq \|\sqrt{\mathrm{L}_A}u\|^2 \leq \Lambda\|\nabla u\|^2$, we obtain that $\mathcal{N}(\nabla) = \mathcal{N}(\sqrt{\mathrm{L}_A})$ and hence, $\mathcal{N}(\nabla) = \mathcal{N}(\mathrm{L}_A)$.

Now, recall that $(\mathcal{M}, \tilde{\mathrm{g}})$ admits a Poincaré inequality: there exists $C > 0$ such that

$$\|u - u_{\mathcal{M}}\| \leq C\|\nabla u\|,$$

for every $u \in \mathrm{W}^{1,2}(\mathcal{M})$. Note then that, if $u \in \mathrm{W}^{1,2}(\mathcal{M})$ and $\int_{\mathcal{M}} u \, d\mu_{\tilde{\mathrm{g}}} = 0$, then $\|u\| \leq C\|\nabla u\|$. Thus, if we further assume that $u \in \mathcal{N}(\nabla)$ then we obtain that $u = 0$. On letting $Z = \left\{u \in \mathrm{L}^2(\mathcal{M}) : \int_{\mathcal{M}} u \, d\mu_{\tilde{\mathrm{g}}} = 0\right\}$, that is precisely

$$\{0\} = \mathcal{N}(\mathrm{L}_A) \cap \mathrm{W}^{1,2}(\mathcal{M}) \cap Z = \mathcal{N}(\mathrm{L}_A) \cap Z,$$

since $\mathcal{N}(\mathrm{L}_A) \subset \mathrm{W}^{1,2}(\mathcal{V})$. Therefore, $Z \subset \mathcal{N}(\mathrm{L}_A)^{\perp} = \overline{\mathcal{R}(\mathrm{L}_A)}$. For the reverse inclusion, let $u \in \overline{\mathcal{R}(\mathrm{L}_A)}$. Then, there exists a sequence $v_n \in \mathcal{D}(\mathrm{L}_A)$ such that $u = \lim_{n\to\infty} \mathrm{L}_A v_n$ in $\mathrm{L}^2(\mathcal{M})$. Then,

$$\begin{aligned}\int_{\mathcal{M}} u \, d\mu_{\tilde{\mathrm{g}}} &= \int_{\mathcal{M}} \lim_{n\to\infty} \mathrm{L}_A v_n \cdot 1 \, d\mu_{\tilde{\mathrm{g}}} = \lim_{n\to\infty} \langle \mathrm{L}_A v_n, 1\rangle \\ &= \lim_{n\to\infty} \langle -\operatorname{div}_{\tilde{\mathrm{g}}} A\nabla v_n, 1\rangle = \lim_{n\to\infty} \langle A\nabla v_n, \nabla(1)\rangle = 0.\end{aligned}$$

□

Next, we note that since L_A preserves the spaces $\mathcal{N}(\mathrm{L}_A)$ and $\overline{\mathcal{R}(\mathrm{L}_A)}$, we obtain that the restricted operator

$$\mathrm{L}_A^R = \mathrm{L}_A|_{\overline{\mathcal{R}(\mathrm{L}_A)}} : \overline{\mathcal{R}(\mathrm{L}_A)} \to \overline{\mathcal{R}(\mathrm{L}_A)},$$

and

$$J_A^R = J_A|_{\overline{\mathcal{R}(\mathrm{L}_A)} \cap \mathrm{W}^{1,2}(\mathcal{M})},$$

with $\mathcal{D}(J_A^R) = \mathrm{W}^{1,2}(\mathcal{M}) \cap \overline{\mathcal{R}(\mathrm{L}_A)}$.

Proposition 4.2. *The operator L_A^R is a closed, densely-defined operator with associated form J_A^R.*

Proof. By definition, we obtain that $\mathcal{D}(\mathrm{L}_A^R) = \mathcal{D}(\mathrm{L}_A) \cap \overline{\mathcal{R}(\mathrm{L}_A)}$. Now, let D_A^R be the operator given via the form J_A^R. We note that J_A^R is both densely-defined and closed. Now, note that

$$\mathcal{D}(D_A^R) = \left\{u \in \overline{\mathcal{R}(\mathrm{L}_A)} : \mathrm{W}^{1,2}(\mathcal{M}) \cap \mathcal{R}(\mathcal{R}(\mathrm{L}_A)) \ni v \mapsto J_A^R[u, v] \text{ continuous}\right\}.$$

It is easy to see that $\mathcal{D}(\mathrm{L}_A) \cap \overline{\mathcal{R}(\mathrm{L}_A)} \subset \mathcal{D}(D_A^R)$.

For the reverse inclusion, assume that $u \in \mathcal{D}(D_A^R)$, and $v \in \mathcal{R}(\mathrm{L}_A) \cap \mathrm{W}^{1,2}(\mathcal{M})$ so that $v \mapsto \langle A\nabla u, \nabla v\rangle$ is continuous. Now, we have that $\mathcal{R}(\mathrm{L}_A)$ is dense in $\mathrm{L}^2(\mathcal{M})$ as well as in $\mathrm{W}^{1,2}(\mathcal{M}) = \mathcal{D}(\sqrt{\mathrm{L}_A})$ (by a functional calculus argument). Thus, this continuity is valid for every $v \in \mathrm{W}^{1,2}(\mathcal{M})$ and hence, $u \in \mathcal{D}(\mathrm{L}_A)$. Since we have by assumption that $u \in \overline{\mathcal{R}(\mathrm{L}_A)}$, we have that $\mathcal{D}(D_A^R) \subset \mathcal{D}(\mathrm{L}_A) \cap \overline{\mathcal{R}(\mathrm{L}_A)}$.

It is easy to see that $\mathrm{L}_A^R = D_A^R$ which is a closed, densely-defined operator by the Lax-Milgram theorem. □

We compute the spectrum of L_A via the spectrum for L_A^R.

Proposition 4.3. *The spectra of the operators* L_A *and* L_A^R *relate by:*

$$\sigma(\mathrm{L}_A) = \{0\} \cup \sigma(\mathrm{L}_A^R) \subset \{0\} \cup [\kappa\lambda_1(\mathcal{M}, \tilde{\mathrm{g}}), \infty).$$

Proof. Fix $\zeta \neq 0$ and $\zeta \in \rho(\mathrm{L}_A^R)$, That is, $\zeta - \mathrm{L}_A^R : \mathcal{D}(\mathrm{L}_A^R) = \overline{\mathcal{R}(\mathrm{L}_A)} \cap \mathcal{D}(\mathrm{L}_A) \to \overline{\mathcal{R}(\mathrm{L}_A^R)}$ is invertible. Thus, for any $u \in \overline{\mathcal{R}(\mathrm{L}_A)} \cap \mathcal{D}(\mathrm{L}_A)$, there exists a $v \in \overline{\mathcal{R}(\mathrm{L}_A)}$ such that $u = (\zeta - \mathrm{L}_A^R)^{-1}v$ or equivalently, $v = (\zeta - \mathrm{L}_A^R)u$. But we have that $\mathrm{L}_A^R = \mathrm{L}_A|_{\overline{\mathcal{R}(\mathrm{L}_A)} \cap \mathcal{D}(\mathrm{L}_A)}$ and therefore, we obtain that $v = (\zeta - \mathrm{L}_A)u$. That is, $u = (\zeta - \mathrm{L}_A^R)^{-1}(\zeta - \mathrm{L}_A)u$. From this, we also obtain that $v = (\zeta - \mathrm{L}_A)u = (\zeta - \mathrm{L}_A)(\zeta - \mathrm{L}_A^R)^{-1}v$. That is, on $\overline{\mathcal{R}(\mathrm{L}_A)}$, $(\zeta - \mathrm{L}_A)^{-1} = (\zeta - \mathrm{L}_A^R)^{-1}$.

Now, fix $u \in \mathcal{N}(\mathrm{L}_A)$. Then, we have that $(\zeta - \mathrm{L}_A)u = \zeta u$. Since we assume $\zeta \neq 0$, we obtain that $(\zeta - \mathrm{L}_A^{-1})u = \zeta^{-1}u$. This proves that $\rho(\mathrm{L}_A^R) \setminus 0 \subset \rho(\mathrm{L}_A)$ or $\sigma(\mathrm{L}_A) \subset \sigma\rho\mathrm{L}_A^R \cup \{0\}$

Now, suppose that $\zeta \in \rho(\mathrm{L}_A)$. Then, for $u \in \overline{\mathcal{R}(\mathrm{L}_A)}$ there exists a $v \in \overline{\mathcal{R}(\mathrm{L}_A)} \cap \mathcal{D}(\mathrm{L}_A)$ such that $u = (\zeta - \mathrm{L}_A)v$. But $(\zeta - \mathrm{L}_A)v = (\zeta - \mathrm{L}_A^R)v$. By the invertibility of $(\zeta - \mathrm{L}_A)$ we obtain the invertibility of $(\zeta - \mathrm{L}_A^R)$. Thus, $\rho(\mathrm{L}_A) \subset \rho(\mathrm{L}_A^R)$ and $\sigma(\mathrm{L}_A^R) \subset \sigma(\mathrm{L}_A)$. Since we already know that $0 \in \sigma(\mathrm{L}_A)$, we obtain that $\sigma(\mathrm{L}_A^R) \cup \{0\} \subset \sigma(\mathrm{L}_A)$.

Next, note that since L_A^R is self-adjoint, $\sigma(\mathrm{L}_A^R) \subset \overline{\mathrm{nr}(\mathrm{L}_A^R)}$ where

$$\mathrm{nr}(\mathrm{L}_A^R) = \left\{ \langle \mathrm{L}_A u, u \rangle : u \in \mathcal{D}(\mathrm{L}_A^R),\ \|u\| = 1 \right\},$$

is the numerical-range of L_A^R. Moreover, via the Poincaré inequality, we obtain that

$$J_A^R[u, u] \geq \kappa \|\nabla u\|^2 \geq \kappa\lambda_1(\mathcal{M}, \tilde{\mathrm{g}})\|u\|^2$$

for $u \in \overline{\mathcal{R}(\mathrm{L}_A)} \cap \mathrm{W}^{1,2}(\mathcal{M})$, and where $\lambda_1(\mathcal{M}, \tilde{\mathrm{g}})$ is the first nonzero eigenvalue for the smooth Laplacian $\Delta_{\tilde{\mathrm{g}}}$. This shows that $\rho(\mathrm{L}_A^R) \subset \mathbb{C} \setminus [\kappa\lambda_1(\mathcal{M}, \tilde{\mathrm{g}}), \infty)$. □

Moreover, we obtain that the operator L_A has discrete spectrum.

Proposition 4.4. *The spectrum* $\sigma(\mathrm{L}_A) = \{0 < \lambda_1 \leq \lambda_2 < \dots\}$ *is discrete and* $\lambda_1 \geq \kappa\lambda_1(\mathcal{M}, \tilde{\mathrm{g}})$.

Proof. Fix $\delta > 0$ and write $\mathrm{L}_{A,\delta}u = \mathrm{L}_A u + \delta u$. It is easy to see that $\sigma(\mathrm{L}_{A,\delta}) = \{\delta\} \cup [\kappa\lambda_1(\mathcal{M}, \tilde{\mathrm{g}}) + \delta, \infty)$. Moreover, the operator $\mathrm{L}_{A,\delta}$ is invertible, in fact, $\mathrm{L}_{A,\delta}^{-1}$ is a resolvent of L_A and thus $\mathrm{L}_{A,\delta}^{-1} : \mathrm{L}^2(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ boundedly.

Furthermore, note that

$$\|\nabla \mathrm{L}_{A,\delta}^{-1}u\| \lesssim \|\sqrt{\mathrm{L}_{A,\delta}}\mathrm{L}_{A,\delta}^{-1}u\| = \|\mathrm{L}_{A,\delta}^{-\frac{1}{2}}u\| \lesssim \|u\|.$$

Thus, we can obtain that $\mathrm{L}_{A,\delta}^{-1} : \mathrm{L}^2(\mathcal{M}) \to \mathrm{W}^{1,2}(\mathcal{M})$ boundedly. Let us call this operator $\mathrm{L}_{A,\delta,\mathrm{W}^{1,2}}^{-1}$.

Now, on a compact manifold, the inclusion map $E : \mathrm{W}^{1,2}(\mathcal{M}) \to \mathrm{L}^2(\mathcal{M})$ is compact. Thus, we can write $\mathrm{L}_{A,\delta}^{-1} = E\mathrm{L}_{A,\delta,\mathrm{W}^{1,2}}^{-1}$. This shows that $\mathrm{L}_{A,\delta}^{-1}$ is compact. That is, $\sigma(\mathrm{L}_{A,\delta}^{-1})$ is discrete with 0 as the only accumulation point. Hence, by combining with our previous bound for the spectrum of $\sigma(\mathrm{L}_A)$, we obtain the claim. □

By the invertibility of L_A^R on $\overline{\mathcal{R}(\mathrm{L}_A)}$, and by our previous characterisation of $\overline{\mathcal{R}(\mathrm{L}_A)}$, we obtain the following first existence result.

Proposition 4.5. *For every $f \in \mathrm{L}^2(\mathcal{M})$ satisfying $\int_{\mathcal{M}} f \; d\mu_{\tilde{\mathrm{g}}} = 0$, we obtain a unique solution $u \in \mathrm{W}^{1,2}(\mathcal{M})$ with $\int_{\mathcal{M}} u \; d\mu_{\tilde{\mathrm{g}}} = 0$ to the equation $\mathrm{L}_A u = f$. This solution is given by $u = (\mathrm{L}_A^R)^{-1} f$.*

Proof. The operator L_A^R is invertible by the fact that the associated form is bounded and coercive. Moreover, it is easy to see that $\mathrm{L}_A^R : \mathcal{D}(\mathrm{L}_A) \cap \overline{\mathcal{R}(\mathrm{L}_A)} \to \overline{\mathcal{R}(\mathrm{L}_A)}$ and hence, $(\mathrm{L}_A^R)^{-1} : \overline{\mathcal{R}(\mathrm{L}_A)} \to \mathcal{D}(\mathrm{L}_A) \cap \overline{\mathcal{R}(\mathrm{L}_A)}$. The uniqueness is by virtue of the fact that $\mathcal{N}(\mathrm{L}_A^R) = \{0\}$. The fact that the solution $u \in \mathrm{W}^{1,2}(\mathcal{M})$ easily follows since $(\mathcal{D}(\mathrm{L}_A), \|\cdot\|_{\mathrm{L}_A}) \subset \mathrm{W}^{1,2}(\mathcal{M})$ is a continuous embedding. □

4.2. Existence and uniqueness for a similar problem.

Let us return to the situation where g is a rough metric on $\mathcal{M}$. Recall that in this situation, there exists a constant $C \geq 1$ and a smooth metric $\tilde{\mathrm{g}}$ that is C-close to g. Let $B \in \mathrm{L}^\infty(\mathcal{T}^{(1,1)}\mathcal{M})$ be such that

$$\mathrm{g}_x(u, v) = \tilde{\mathrm{g}}_x(B(x)u, v)$$

for almost-every $x \in \mathcal{M}$. Let $\theta(x) = \sqrt{\det B(x)}$ for almost-every $x \in \mathcal{M}$ so that $\mathrm{g}(x) = \theta(x) d\mu_{\tilde{\mathrm{g}}}(x)$.

For the sake of convenience, we write $\mathrm{D}_x = -\operatorname{div}_{\mathrm{g}} \omega_x \nabla$. It is easy to see that $J_x[u, v] = \langle \omega_x \nabla u, \nabla v \rangle_{\mathrm{g}}$ is the symmetric form associated to D_x.

First, we note the following.

Proposition 4.6. *There exist $\kappa > 0$ such that $J_x[u, u] \geq \kappa \|\nabla u\|_{\mathrm{g}}^2$ uniformly for $x \in \mathcal{M}$. Moreover, $J_x[u, v] = \langle \omega_x B\theta \nabla u, \nabla v \rangle_{\tilde{\mathrm{g}}}$ and $J_x[u, u] \geq \kappa C^{1+\frac{n}{4}} \|\nabla u\|_{\tilde{\mathrm{g}}}$. A function $\varphi \in \mathrm{W}^{1,2}(\mathcal{M})$ solves (F) if and only if*

$$-\operatorname{div}_{\tilde{\mathrm{g}}}(\theta B \omega_x \nabla \varphi) = \theta \eta.$$

Proof. First, we note that $\omega(x, y) > 0$ for all x, y. Secondly, since we assume that $\omega \in \mathrm{C}^{0,1}(\mathcal{M}^2)$, for $k \geq 1$, by the virtue of compactness of $\mathcal{M}$, we obtain that $\inf_{x,y} \omega(x, y) = \min_{x,y} \omega(x, y) > 0$. That is, set $\kappa = \min_{x,y} \omega(x, y)$ and we're done.

The description of J_x in $\tilde{\mathrm{g}}$ and its ellipticity estimate in $\tilde{\mathrm{g}}$ follow from Proposition 2.10 (i) with $p = 2$, and $r = 1$, $s = 0$.

The equivalence of solutions for the (F) simply follows from Proposition 2.10 (vi). ☐

This is the crucial observation which allows us to reduce the (CE) to solving a divergence form equation with bounded, measurable coefficients on the nearby smooth metric $\tilde{\mathrm{g}}$.

With the aid of this, we demonstrate existence and uniqueness of solutions to (CE).

Proposition 4.7. *Let $\eta \in \mathrm{L}^2(\mathcal{M})$ satisfy $\int_{\mathcal{M}} \eta \, d\mu_{\mathrm{g}} = 0$. Then, there exists a unique solution $\varphi \in \mathrm{W}^{1,2}(\mathcal{M})$ satisfying (F) such that $\int_{\mathcal{M}} \varphi \, d\mu_{\mathrm{g}} = 0$.*

Proof. Let $f = \theta\eta$. Then, note that

$$\int_{\mathcal{M}} f \, d\mu_{\tilde{\mathrm{g}}} = \int_{\mathcal{M}} \eta\theta \, d\mu_{\tilde{\mathrm{g}}} = \int_{\mathcal{M}} \eta \, d\mu_{\mathrm{g}} = 0.$$

Set $A = \theta B \omega_x$ and by what we have proved about f, we are able to apply Proposition (4.5) to the operator L_A in $\langle \cdot , \cdot \rangle_{\tilde{\mathrm{g}}}$ to obtain a unique solution $\tilde{\varphi}$ satisfying $\mathrm{L}_A \tilde{\varphi} = f = \theta\eta$ with $\int_{\mathcal{M}} \tilde{\varphi} \, d\mu_{\tilde{\mathrm{g}}} = 0$.

Define $\varphi(y) = \tilde{\varphi}(y) - \fint_{\mathcal{M}} \tilde{\varphi(y)} \, d\mu_{\mathrm{g}}$ which satisfies $J[\varphi, f] = \langle \mathrm{d}_x(\rho_t^{\mathrm{g}}(x, \cdot), f \rangle_{\tilde{\mathrm{g}}}$ and we also find that

$$\int_{\mathcal{M}} \varphi(y) \, d\mu_{\mathrm{g}}(y) = \int_{\mathcal{M}} \tilde{\varphi}(y) \, d\mu(y) - \int_{\mathcal{M}} (\fint_{\mathcal{M}} \tilde{\varphi}(y) \, d\mu_{\mathrm{g}}(y)) \, d\mu_{\mathrm{g}}(y) = 0.$$

Thus, φ solves (F).

To prove uniqueness, let us fix two solutions φ^1 and φ^2 solving (F) with $\int_{\mathcal{M}} \varphi^i \, d\mu_{\mathrm{g}} = 0$. Then, on writing $\psi = \varphi^1 - \psi^2$, we obtain that ψ satisfies

$$-\operatorname{div}_{\mathrm{g}} \omega_x \nabla \psi = 0$$

with $\int_{\mathcal{M}} \psi \, d\mu = 0$. Now, define $\tilde{\psi}(y) = \psi(y) - \fint_{\mathcal{M}} \psi d\mu_{\tilde{\mathrm{g}}}$. It is easy to see that $\tilde{\psi}$ satisfies

$$-\mathrm{L}_A \tilde{\psi} = 0.$$

with $\int_{\mathcal{M}} \tilde{\psi} \, d\mu_{\tilde{\mathrm{g}}} = 0$. Thus, by the uniqueness guaranteed by Proposition 4.5, we obtain that $\tilde{\psi} = 0$. That is, $\psi(y) = \fint_{\mathcal{M}} \psi(y) \, d\mu_{\tilde{\mathrm{g}}}(y)$, and on integrating this with respect to g, we obtain that

$$0 = \mu_{\mathrm{g}}(\mathcal{M}) \fint_{\mathcal{M}} \psi(y) \, d\mu_{\tilde{\mathrm{g}}}(y).$$

That is, $\psi = \tilde{\psi} = 0$. ☐

4.3. **Operator theory of $x \mapsto \mathrm{D}_x$.** Let us return to the PDE (F), and recall the operator $\mathrm{D}_x = -\operatorname{div}_{\mathrm{g}} \omega_x \nabla$ where $(x, y) \mapsto \omega_x(y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$, and $x \mapsto \omega_x \in \mathrm{C}^{k,\alpha}(\mathcal{N}^2)$. Let $J_x[u, v]$ be its associated symmetric form, $J_x[u, v] = \langle \omega_x \nabla u, \nabla v \rangle$.

In order to understand the regularity $x \mapsto \varphi_{t,x,v}$ of solutions to (CE), we need to prove some preliminary regularity results about the operator family D_x. First, we obtain the constancy of domain as well as the following formula.

Proposition 4.8. *The family of operators* $\mathcal{M} \ni x \mapsto \mathrm{D}_x$ *satisfies* $\mathcal{D}(\mathrm{D}_x) = \mathcal{D}(\Delta_{\mathrm{g}})$ *and* $\mathrm{D}_x u = \omega_x \Delta_{\mathrm{g}} u - \mathrm{g}(\nabla u, \nabla \omega_x)$.

Proof. Fix $x \in \mathcal{M}$ and $u, v \in \mathrm{W}^{1,2}(\mathcal{M})$. Then, note that $\omega_x v \in \mathrm{W}^{1,2}(\mathcal{M})$ and that,

$$\langle \nabla u, \nabla(\omega_x v)\rangle_{\mathrm{g}} = \langle \nabla u, \omega_x \nabla v + v \nabla \omega_x\rangle_{\mathrm{g}} = \langle \omega_x \nabla u, \nabla v\rangle_{\mathrm{g}} + \langle \mathrm{g}(\nabla u, \nabla \omega_x), v\rangle_{\mathrm{g}}.$$

That is, $\langle \omega_x \nabla u, \nabla v\rangle_{\mathrm{g}} = \langle \nabla u, \nabla(\omega_x v)\rangle_{\mathrm{g}} - \langle \mathrm{g}(\nabla u, \nabla \omega_x), v\rangle_{\mathrm{g}}$.

First we show that for any $u \in \mathrm{W}^{1,2}(\mathcal{M})$, $v \mapsto \langle \mathrm{g}(\nabla u, \nabla \omega_x), v\rangle_{\mathrm{g}}$ is continuous. Observe that $|\langle \mathrm{g}(\nabla u, \nabla \omega_x), v\rangle_{\mathrm{g}}| \leq \|\mathrm{g}(\nabla u, \nabla \omega_x)\|_{\mathrm{g}} \|v\|_{\mathrm{g}}$ by the Cauchy-Schwarz inequality. Moreover,

$$\|\mathrm{g}(\nabla u, \nabla \omega_x)\|_{\mathrm{g}}^2 = \int_{\mathcal{M}} |\mathrm{g}(\nabla u, \nabla \omega_x)|^2 \, d\mu_{\mathrm{g}} \leq \int_{\mathcal{M}} |\nabla u|^2 |\nabla \omega_x|^2 \, d\mu_{\mathrm{g}}.$$

However, since $y \mapsto \omega_x(y) \in \mathrm{C}^{0,1}(\mathcal{M})$ and $\mathcal{M}$ is compact, we have that $\operatorname{esssup}_y |\nabla \omega_x(y)| \leq C$, and hence, $\|\mathrm{g}(\nabla u, \nabla \omega_x)\|_{\mathrm{g}} \leq C \|\nabla u\|_{\mathrm{g}}$. This proves that $v \mapsto \langle \mathrm{g}(\nabla u, \nabla \omega_x), v\rangle_{\mathrm{g}}$ is continuous.

Now, suppose that $u \in \mathcal{D}(\Delta_{\mathrm{g}})$, then $\langle \nabla u, \nabla(\omega_x v)\rangle_{\mathrm{g}} = \langle \Delta_{\mathrm{g}} u, \omega_x v\rangle$ and hence, $v \mapsto \langle \nabla u, \nabla(\omega_x v)\rangle_{\mathrm{g}}$ is continuous. Since we have already shown that $v \mapsto \langle \mathrm{g}(\nabla u, \nabla \omega_x), v\rangle_{\mathrm{g}}$ is continuous, we obtain that $v \mapsto \langle \omega_x \nabla u, \nabla v\rangle$ is continuous. Hence, $u \in \mathcal{D}(\mathrm{D}_x)$ which proves that $\mathcal{D}(\Delta_{\mathrm{g}}) \subset \mathcal{D}(\mathrm{D}_x)$.

Similarly, for $u \in \mathcal{D}(\mathrm{D}_x)$, we find that $v \mapsto \langle \nabla u, \nabla(\omega_x v)\rangle_{\mathrm{g}}$ is continuous. Hence, $u \in \mathcal{D}(\omega_x \Delta_{\mathrm{g}}) = \mathcal{D}(\Delta_{\mathrm{g}})$. This shows that $\mathcal{D}(\mathrm{D}_x) \subset \mathcal{D}(\Delta_{\mathrm{g}})$. □

Remark 4.9. *We note that an immediate consequence of this is that the unique solution* φ *to* (F) *satisfies* $\varphi \in \mathcal{D}(\Delta_{\mathrm{g}})$, *since* $\mathcal{D}(\Delta_{\mathrm{g}}) = \mathcal{D}(\mathrm{D}_x) = \mathcal{D}(\mathrm{L}_x)$. *This observation is essential to obtain regularity of solutions.*

We also obtain the following uniform boundedness for the operator family parametrised in $x \in \mathcal{N}$.

Proposition 4.10. *The family of operators* $\mathcal{M} \ni x \mapsto \mathrm{D}_x : (\mathcal{D}(\Delta_{\mathrm{g}}), \|\cdot\|_{\Delta_{\mathrm{g}}}) \to \mathrm{L}^2(\mathcal{M})$ *is a uniformly bounded family of operators. Moreover,* $\|u\|_{\mathrm{D}_x} \simeq \|u\|_{\Delta_{\mathrm{g}}}$ *holds with the implicit constant independent of* $x \in \mathcal{M}$.

Proof. We show that $\|\mathrm{D}_x u\| \lesssim \|u\|_{\Delta_{\mathrm{g}}} = \|\Delta_{\mathrm{g}} u\| + \|u\|$, where the implicit constant is independent of $x \in \mathcal{M}$. Fix $x \in \mathcal{M}$, and note that

$$\begin{aligned}\|\mathrm{D}_x u\| &\leq \|\omega_x \Delta_{\mathrm{g}} u\| + \|\mathrm{g}(\nabla u, \nabla \omega_x)\| \\ &\leq \left(\sup_{y \in \mathcal{M}} |\omega_x(y)|\right) \|\Delta_{\mathrm{g}} u\| + \left(\operatorname{esssup}_{y \in \mathcal{M}} |\nabla \omega_x(y)|\right) \|\nabla u\|.\end{aligned}$$

But since $(x, y) \mapsto \omega_x(y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$, the Lipschitz constant is in both variables, and hence, there is a $C > 0$ such that $\operatorname{esssup}_{y \in \mathcal{M}} |\nabla \omega_x(y)| \leq C$. The quantity $\sup_{y \in \mathcal{M}} |\omega_x(y)|$ is also independent of C simply by coupling the continuity of $(x, y) \mapsto \omega_x(y)$ with the compactness of $\mathcal{M}$.

Now, note that by ellipticity,

$$\|\nabla u\|^2 \leq |\langle \Delta_{\mathrm{g}} u, u\rangle| \leq \|\Delta_{\mathrm{g}} u\|\|u\| \leq \|\Delta_{\mathrm{g}} u\|^2 + \|u\|^2.$$

This complete the proof. The reverse inequality is argued similarly on noting that $\omega_x(y) > 0$ for all $x, y \in \mathcal{M}$. □

Remark 4.11. *Note that the two previous propositions are valid on all of $\mathcal{M}$, not just on $\mathcal{N} \subset \mathcal{M}$ where $x \mapsto \omega_x$ enjoys higher regularity.*

Let $v \in \mathrm{T}_x\mathcal{M}$ and $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Let $f : \mathcal{N} \to \mathcal{V}$, where $\mathcal{V}$ where $\mathcal{V}$ is some normed vector space. Then, we write the difference quotient as

$$Q_s^v f(x) = \frac{f(x) - f(\gamma(s))}{s}.$$

We define the *directional derivative* of f (when it exists, and it is independent of the generating curve γ) to be

$$(\mathrm{d}_x f(x))(v) = \lim_{s\to 0} Q_s^v f(x).$$

In our particular setting, we consider $\mathcal{V} = \mathrm{L}^2(\mathcal{M})$ with the weak topology for the choice $f(x) = \mathrm{D}_x$. More precisely, if there exists $\tilde{\mathrm{D}}_x : \mathcal{D}(\Delta_{\mathrm{g}}) \to \mathrm{L}^2(\mathcal{M})$ satisfying

$$\lim_{s\to 0} \langle Q_s^v \mathrm{D}_x u, w\rangle = \langle \tilde{\mathrm{D}}_x u, w\rangle,$$

for every $w \in \mathrm{W}^{1,2}(\mathcal{M})$, we say that D_x has a *(weak) derivative* at x and write $(\mathrm{d}_x \mathrm{D}_x) = \tilde{\mathrm{D}}_x$. In what is to follow, we will see that this is a sufficiently strong enough notion of derivative to obtain regularity properties for the flow defined by (GM).

With this notation at hand, we prove the following important proposition.

Proposition 4.12. *The operator valued function $\mathcal{N} \ni x \mapsto \mathrm{D}_x : \mathcal{D}(\Delta_{\mathrm{g}}) \to \mathrm{L}^2(\mathcal{M})$ is weakly differentiable k times. At each $x \in \mathcal{N}$ and for every $v \in \mathrm{T}_x\mathcal{M}$,*

$$(\mathrm{d}_x \mathrm{D}_x)(v) = -\operatorname{div}_{\mathrm{g}}((\mathrm{d}_x \omega_x)(v))\nabla : \mathcal{D}(\Delta_{\mathrm{g}}) \to \mathrm{L}^2(\mathcal{M})$$

is densely-defined and symmetric. Moreover, inside a chart $\Omega \Subset \mathcal{N}$ containing x for which the vector v is constant, there is a constant C_Ω such that

$$\|(\mathrm{d}_x \mathrm{D}_x)(v) u\| \leq C_\Omega \|u\|_{\Delta_{\mathrm{g}}}.$$

Proof. Fix $x \in \mathcal{N}$ and a chart $\Omega \Subset \mathcal{N}$ with a constant vector $v \in \mathrm{T}_x\mathcal{M}$, and note that because of the higher regularity of ω_x at x, we have $(x, y) \mapsto (\mathrm{d}_x \omega_x(y))(v) \in \mathrm{C}^0(\Omega \times \mathcal{M})$ and $x \mapsto (\mathrm{d}_x \omega_x(y))(v) \in \mathrm{C}^{0,1}(\mathcal{M})$. Coupling this with the compactness of $\mathcal{M}$ and $\overline{\Omega}$, there exists $\Lambda > 0$ such that $-\Lambda < (\mathrm{d}_x \omega_x(y))(v) < \Lambda$, for all $x \in \Omega$ and $y \in \mathcal{M}$. Thus, let $f_{x,\varepsilon} = (\mathrm{d}_x \omega_x)(v) + \Lambda + \varepsilon$ and define $K_\varepsilon[u, w] = \langle f_{x,\varepsilon} \nabla u, \nabla w\rangle$. By the Lax-Milgram theorem, the operator associated to the form K_ε is exactly

$$\tilde{\mathrm{D}}_{x,\varepsilon} = -\operatorname{div}_{\mathrm{g}}[((\mathrm{d}_x \omega_x)(v)) + (\Lambda + \varepsilon)]\nabla,$$

and is guaranteed to be non-negative self-adjoint. Since we have that ω_x is k times differentiable at our chosen x, the map $y \mapsto f_{x,\varepsilon}(y)$ is still Lipschitz and hence, we are able to apply Proposition 4.8 with $f_{x,\varepsilon}$ in place of ω_x to obtain that $\mathcal{D}(\tilde{\mathrm{D}}_{x,\varepsilon}) = \mathcal{D}(\Delta_{\mathrm{g}})$. Consequently, we obtain that $\tilde{\mathrm{D}}_{x,\varepsilon} - (\Lambda+\varepsilon)\Delta_{\mathrm{g}}$ has domain $\mathcal{D}(\Delta_{\mathrm{g}})$ and so, an easy calculation via the defining form, demonstrates that

$$\mathrm{D}'_x u = -\operatorname{div}_{\mathrm{g}}(\mathrm{d}_x\omega_x)(v)\nabla u = \tilde{\mathrm{D}}_{x,\varepsilon}u - (\Lambda+\varepsilon)\Delta_{\mathrm{g}}u,$$

from which its clear that the operator is densely-defined.

A repetition of the argument in Proposition 4.10, utilising the higher regularity of $x \mapsto \omega_x$ on $\mathcal{N}$, there exists a constant $C_\Omega > 0$ such that $|(\mathrm{d}_x\omega_x(y))(v)| \leq C_\Omega$ for all $x \in \Omega$ and almost-every $y \in \mathcal{M}$. Thus, $\|\mathrm{D}_{x,\varepsilon}u\| \leq C_\Omega \|u\|_{\Delta_{\mathrm{g}}}$ and the estimate in the conclusion follows.

Now we show that the formula in the conclusion is valid. Fix $w \in \mathrm{W}^{1,2}(\mathcal{M})$, $u \in \mathcal{D}(\Delta_{\mathrm{g}})$ and compute

$$\begin{aligned}\lim_{s\to 0}\langle Q^v_s \mathrm{D}_x u, w\rangle = \lim_{s\to 0}\langle -\operatorname{div} Q^v_s\omega_x \nabla u, w\rangle &= \lim_{s\to 0}\langle Q^v_s \omega_x\nabla u, \nabla w\rangle \\ &= \lim_{s\to 0}\int_{\mathcal{M}} Q^v_s\omega_x(y)\mathrm{g}_y(\nabla u(y),\nabla w(y))\, d\mu_{\mathrm{g}}(y).\end{aligned}$$

Now, note that

$$|Q^v_s\omega_x(y)| \leq \left|\frac{\omega_x(y)-\omega_{\gamma(s)}(y)}{s}\right| \leq C$$

since $(x,y) \mapsto \omega_x(y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$ and $x \mapsto \omega_x \in \mathrm{C}^1(\mathcal{M})$. So, we are able to apply the dominated convergence theorem to obtain

$$\begin{aligned}\lim_{s\to 0}\langle Q^v_s \mathrm{D}_x u, w\rangle &= \int_{\mathcal{M}} \lim_{s\to 0} Q^s_v \omega_x(y)\mathrm{g}_y(\nabla u(y),\nabla w(y))\, d\mu_{\mathrm{g}}(y) \\ &= \langle (\mathrm{d}_x\mathrm{D}_x)(v)\nabla u, \nabla w\rangle = \langle -\operatorname{div}_{\mathrm{g}}(\mathrm{d}_x\mathrm{D}_x)(v)\nabla u, w\rangle,\end{aligned}$$

where the last equality follows from the fact that we assume that $u \in \mathcal{D}(\Delta_{\mathrm{g}})$ and we have already shown that $\mathcal{D}(-\operatorname{div}_{\mathrm{g}}((\mathrm{d}_x\omega_x)(v))\nabla) = \mathcal{D}(\Delta_{\mathrm{g}})$.

The equality of operators in the conclusion follows from the fact that $w \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$ is dense in $\mathrm{L}^2(\mathcal{M})$. □

Remark 4.13. *Let $v_1,\dots,v_l \in \mathrm{T}_x\mathcal{M}$, with $k \leq l$, and note that the map $(x,y) \mapsto (\mathrm{d}^l_x\omega_x)(v_1,\dots,v_l) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$, where $(\mathrm{d}^2_x\omega_x)(v_1,v_2) = (\mathrm{d}_x(\mathrm{d}_x\omega_x)(v_1))(v_2)$. Thus, on applying this proposition repeatedly, we can assert that*

$$(\mathrm{d}^l_x\mathrm{D}_x)(v_1,\dots,v_l) = -\operatorname{div}_{\mathrm{g}}((\mathrm{d}^l_x\omega_x)(v_1,\dots,v_l))\nabla : \mathcal{D}(\Delta_{\mathrm{g}}) \to \mathrm{L}^2(\mathcal{M})$$

is a densely-defined operator.

Now, we are able to prove the following product rule for the operator D_x. This product rule is the essential tool for obtaining the existence of weak derivatives $\partial_x\varphi_x$ of solutions for (CE).

Proposition 4.14. *Let $x \mapsto u_x : \mathcal{N} \to \mathcal{D}(\Delta_{\mathrm{g}})$, $v \in \mathrm{T}_x\mathcal{M}$ and suppose that $(\mathrm{d}_x u_x)(v)$ exists weakly. Then $(\mathrm{d}_x\mathrm{D}_x u_x)(v)$ exists weakly if and only if $\mathrm{D}_x((\mathrm{d}_x u_x)(v))$ exists weakly and*

$$(\mathrm{d}_x\mathrm{D}_x u_x)(v) = (\mathrm{d}_x\mathrm{D}_x)(v)u_x + \mathrm{D}_x((\mathrm{d}_x u_x)(v)).$$

Proof. Fix $w \in \mathrm{W}^{1,2}(\mathcal{M})$ and define $f(x,y) = \langle \mathrm{D}_x u_y, w\rangle$. Let $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ be a curve in $\mathcal{M}$ satisfying $\gamma(0) = x$ and $\dot\gamma(0) = v$. Now, note that for $s > 0$,

$$(\dagger)\quad \frac{1}{s}[f(\gamma(s),\gamma(s)) - f(x,x)] = \frac{1}{s}[f(\gamma(s),\gamma(s)) - f(x,\gamma(s))] + \frac{1}{s}[f(x,\gamma(s)) - f(x,x)]$$

By Proposition 4.12, $(\mathrm{d}_x f(x,y))(v)|_{y=x}$ exists and

$$\begin{aligned}(\mathrm{d}_x f(x,y))(v)|_{y=x} &= \lim_{s\to 0}\frac{1}{s}[f(\gamma(s),y) - f(x,y)]|_{y=x}\\ &= \lim_{t\to 0}\lim_{s\to 0}\frac{1}{s}[f(\gamma(s),\gamma(t)) - f(x,\gamma(t))]\\ &= \lim_{s\to 0}\frac{1}{s}[f(\gamma(s),\gamma(s)) - f(x,\gamma(s))].\end{aligned}$$

Assume that $(\mathrm{d}_x \mathrm{D}_x u_x)(v)$ exists weakly. Then, by (†), $\lim_{s\to 0} s^{-1}[f(x,\gamma(s)) - f(x,x)]$ exists and so, by further choosing $w \in \mathcal{D}(\Delta_{\mathrm{g}})$,

$$\begin{aligned}\lim_{s\to 0}\frac{1}{s}[f(x,\gamma(s)) - f(x,x)] &= \lim_{s\to 0}\langle \frac{1}{s}\mathrm{D}_x(u_x - u_{\gamma(s)}), w\rangle\\ &= \lim_{s\to 0}\langle Q^v_s u_x, \mathrm{D}_x w\rangle = \langle (\mathrm{d}_x u_x)(v), \mathrm{D}_x w\rangle,\end{aligned}$$

Also, by (†),

$$\langle (\mathrm{d}_x u_x)(v), \mathrm{D}_x w\rangle = \langle \partial_x(\mathrm{D}_x u_x)(v) - (\partial_x \mathrm{D}_x)(v) u_x, w\rangle,$$

and since the right hand side is continuous in w, we obtain that $(\mathrm{d}_x u_x)(v) \in \mathcal{D}(\mathrm{D}_x) = \mathcal{D}(\Delta_{\mathrm{g}})$.

Now, if $\mathrm{D}_x(\mathrm{d}_x u_x)(v)$ exists weakly, then from (†), we are able to assert that the limit $\lim_{s\to 0} s^{-1}[f(\gamma(s),\gamma(s)) - f(x,x)]$ exists, which is precisely that $\mathrm{d}_x(\mathrm{D}_x u_x)(v)$ exists weakly. The product rule formula is obvious from these computations. □

Remark 4.15. *If the function $(x,y) \mapsto \omega_x(y) \in \mathrm{C}^k(\mathcal{M}^2)$ for $k \geq 1$, then we are able to perform this analysis in the uniform operator topology $\mathcal{L}((\mathcal{D}(\Delta_{\mathrm{g}}), \|\cdot\|_{\Delta_{\mathrm{g}}}), \mathrm{L}^2(\mathcal{M}))$. This involves estimating the term $\sup_{x,y\in\mathcal{M}} |\nabla Q^v_s w_x(y)|$ and showing that this quantity tends to 0 as $s \to 0$. It is clear that such an estimate cannot be made even with the supremum replaced by an essential supremum when $(x,y) \mapsto \omega_x(y)$ is only Lipschitz.*

4.4. **Regularity of solutions.** We combine the results obtained in the previous subsections to prove the following regularity theorem for solutions to (F). First, we note the following lemma.

Lemma 4.16. *Let $\int_{\mathcal{M}} u\ d\mu_{\mathrm{g}} = 0$. Then, $\|\mathrm{L}_x^{-\frac{1}{2}} u\| \lesssim \|u\|$ and $\|\mathrm{L}_x^{-1} u\| \lesssim \|u\|$, where the implicit constants are independent of x.*

Proof. For this, note that $\mathrm{D}_x = \theta^{-1}\mathrm{L}_x$ and that

$$\langle \mathrm{L}_x v, v\rangle_{\tilde{\mathrm{g}}} \geq \kappa \|\nabla v\|_{\tilde{\mathrm{g}}}.$$

If we assume that $\int_{\mathcal{M}} v\ d\mu_{\tilde{\mathrm{g}}} = 0$, then we have by Proposition 4.6 that $\|\sqrt{\mathrm{L}_x} v\| \geq \kappa\lambda_1(\mathcal{M}, \tilde{\mathrm{g}})\|v\|$ uniformly in x. This shows the uniform boundedness for $\mathrm{L}_x^{-\frac{1}{2}}$.

Next, set $v = \sqrt{\mathrm{L}_x} w$ for $w \in \mathcal{D}(\Delta_{\mathrm{g}})$, we obtain that $\|\mathrm{L}_x w\|_{\tilde{\mathrm{g}}} = \kappa^2 \lambda_1(\mathcal{M}, \tilde{\mathrm{g}})^2 \|w\|_{\tilde{\mathrm{g}}}$. That is, $\|\mathrm{L}_x^{-1} w\|_{\tilde{\mathrm{g}}} \lesssim \|w\|_{\tilde{\mathrm{g}}}$. Now, for $u \in \mathcal{D}(\Delta_{\mathrm{g}})$ satisfying $\int_{\mathcal{M}} u \; d\mu_{\mathrm{g}} = 0$, we have that $w = \theta u$ satisfies $\int_{\mathcal{M}} w \; d\mu_{\tilde{\mathrm{g}}} = 0$ and $\mathrm{L}_x^{-1} w = \mathrm{L}_x^{-1} u$ and so $\|\mathrm{L}_x^{-1} u\|_{\mathrm{g}} \simeq \|\mathrm{L}_x^{-1} w\|_{\tilde{\mathrm{g}}} \lesssim \|w\|_{\tilde{\mathrm{g}}} = \|\theta u\|_{\tilde{\mathrm{g}}} = \|u\|_{\mathrm{g}}$. □

With this tool in hand, we present the following regularity theorem.

Theorem 4.17. *Suppose that $k \geq 1$ and $(x, y) \mapsto \omega_x(y) \in \mathrm{C}^{0,1}(\mathcal{M}^2)$ and $x \mapsto \omega_x \in \mathrm{C}^k(\mathcal{N})$. Moreover, suppose that $(x, y) \mapsto \eta_x(y) \in \mathrm{C}^0(\mathcal{N} \times \mathcal{M})$ and $x \mapsto \eta_x(y) \in \mathrm{C}^l(\mathcal{N})$ where $l \geq 1$. If at $x \in \mathcal{N}$, φ_x solves (F) with $\int_{\mathcal{M}} \varphi_x \; d\mu_{\mathrm{g}} = \int_{\mathcal{M}} \eta_x \; d\mu_{\mathrm{g}} = 0$, the map $x \mapsto \langle \eta_x, \varphi_x \rangle \in \mathrm{C}^{\min\{k,l\}-1,1}(\mathcal{N})$.*

Proof. Fix $v \in \mathrm{T}_x\mathcal{M}$ and assuming $(\mathrm{d}_x \varphi_x)(v)$ exists in $\mathcal{D}(\Delta_{\mathrm{g}})$, we obtain that $(\mathrm{d}_x \eta_x)(v) = (\mathrm{d}_x \mathrm{D}_x)(v) \varphi_x + \mathrm{D}_x (\mathrm{d}_x \varphi_x)(v)$. Thus, on writing

$$\eta'_{x,v} = (\mathrm{d}_x \eta_x)(v) - (\mathrm{d}_x \mathrm{D}_x)(v) \varphi_x$$

and rearranging the previous expression, we note that $(\mathrm{d}_x \varphi_x)(v)$ exists as a solution to $\mathrm{D}_x (\mathrm{d}_x \varphi_x)(v) = \eta'_x$. We further note that this PDE is again of the form (F).

Fix a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Then,

$$\int_{\mathcal{M}} (\mathrm{d}_x \eta_x)(v) \; d\mu_{\mathrm{g}} = \int_{\mathcal{M}} \frac{d}{dt}|_{t=0} \eta_{\gamma(t)} \; d\mu_{\mathrm{g}} = \frac{d}{dt}|_{t=0} \int_{\mathcal{M}} \eta_{\gamma(t)} \; d\mu_{\mathrm{g}} = 0$$

simply by virtue of the fact that $\int_{\mathcal{M}} \eta_x \; d\mu_{\mathrm{g}} = 0$ for each x.

Next, note that by Proposition 4.12

$$\int_{\mathcal{M}} (\mathrm{d}_x \mathrm{D}_x)(v) \varphi_x \; d\mu_{\mathrm{g}} = \langle (\mathrm{d}_x \mathrm{D}_x)(v) \varphi_x, 1 \rangle_{\mathrm{g}} = \langle (\mathrm{d}_x \omega_x)(v) \nabla \varphi_x, \nabla(1) \rangle_{\mathrm{g}} = 0.$$

Thus, we have shown that $\int_{\mathcal{M}} \eta'_x \; d\mu_{\mathrm{g}} = 0$ and hence, by Proposition 4.7, we obtain that $(\mathrm{d}_x \varphi_x)(v) \in \mathcal{D}(\Delta_{\mathrm{g}}) \subset \mathrm{W}^{1,2}(\mathcal{M})$ exists.

Next, we show that $x \mapsto \langle \eta_x, \varphi_x \rangle$ is differentiable. For that, let us write $f(x, y) = \langle \eta_x, \varphi_y \rangle$. By (†), we have that

$$(\mathrm{d}_x f(x, x))(v) = (\mathrm{d}_x f(x, y))(v)|_{y=x} + \mathrm{d}_x (f(y, x))(v)|_{y=x}$$

when the limits exist.

So, first for the first expression on the left hand side,

$$(\mathrm{d}_x f(x, y))(v)|_{y=x} = \lim_{s \to 0} \langle Q_s^v \eta_x, \varphi_x \rangle = \langle (\mathrm{d}_x \eta_x)(v), \varphi_x \rangle.$$

For the second expression,

$$(\mathrm{d}_x f(y, x))|_{y=x}) = \langle \eta_x, (\mathrm{d}_x \varphi_x)(v) \rangle.$$

Now we show that the directional derivative is bounded in small neighbourhoods containing x. So, fix $\Omega \Subset \mathcal{N}$ a coordinate chart containing x in which the vector v

is constant in this chart. We note that it suffices to show that

$$|(\mathrm{d}_x\langle \eta_x, \varphi_x\rangle)(v)| \lesssim p(\|\eta_x\|, \|(\mathrm{d}_x\eta_x)(v)\|),$$

for a polynomial p since $\|\eta_x\|$, $\|(\mathrm{d}_x\eta_x)(v)\| \leq C$, where the constant C and the implicit constant depend on Ω. This demonstrates that $x \mapsto \langle \eta_x, \varphi_x\rangle$ is continuous at x with bounded derivatives, which in turn implies that this function is Lipschitz, and moreover that the differential exists almost-everywhere.

Recall that by Proposition 4.7, $\varphi_x = \mathrm{L}_x^{-1}\theta\eta_x + c$, where $c_x = \fint_{\mathcal{M}} \mathrm{D}_x^{-1}\theta\eta_x \, d\mu_{\mathrm{g}}$. Similarly, $(\mathrm{d}_x\varphi_x)(v) = \mathrm{L}_x^{-1}\theta\eta'_{x,v} + c'$ where $c'_x = \fint_{\mathcal{M}} \mathrm{L}_x^{-1}\theta\eta'_{x,v}\, d\mu_{\mathrm{g}}$. Hence,

$$|(\mathrm{d}_x f(x,y))(v)|_{y=x}| = |\langle \eta_x, \mathrm{L}_x^{-1}\theta\eta_x + c_x\rangle| \leq \|\eta_x\|\|\mathrm{L}_x^{-1}\theta\eta_x\| + \|c_x\|\|\theta\eta_x\|.$$

Now, note that by Lemma 4.16, $\|\mathrm{L}_x^{-1}\theta\eta_x\| \lesssim \|\eta_x\|$, where the constant is uniform in $x \in \mathcal{M}$, and that

$$\fint_{\mathcal{M}} \mathrm{L}_x^{-1}\theta\eta_x \, d\mu_{\mathrm{g}} \leq \|\mathrm{L}_x^{-1}\theta\eta_x\| \lesssim \|\eta_x\|,$$

which shows that $\|c_x\| \lesssim \|\eta_x\|$. Thus,

$$|(\mathrm{d}_x f(x,y))(v)|_{y=x}| \lesssim \|\eta_x\|^2$$

with the constant independent of $x \in \mathcal{M}$.

We estimate the remaining term,

$$\begin{aligned} |(\mathrm{d}_x f(y,x))|_{y=x})| &= |\langle \eta_x, \mathrm{L}_x^{-1}\theta(\mathrm{d}_x\eta_x)(v) - \mathrm{L}_x^{-1}\theta(\mathrm{d}_x\mathrm{L}_x)(v)\varphi_x) + c'_x\rangle| \\ &\leq |\langle \eta_x, \mathrm{L}_x^{-1}\theta(\mathrm{d}_x\eta_x)(v)\rangle| + |\langle \eta_x, \mathrm{L}_x^{-1}\theta(\mathrm{d}_x\mathrm{L}_x)(v)\varphi_x)\rangle| + |\langle c'_x, \eta_x\rangle|. \end{aligned}$$

Now,

$$|\langle \eta_x, \mathrm{L}_x^{-1}\theta(\mathrm{d}_x\eta_x)(v)\rangle| \lesssim \|\eta_x\| + \|\mathrm{L}_x^{-1}\theta(\mathrm{d}_x\eta_x)(v)\| \lesssim \|\eta_x\| + \|(\mathrm{d}_x\eta_x)(v)\|,$$

and

$$\begin{aligned} |\langle \eta_x, \mathrm{L}_x^{-1}\theta(\mathrm{d}_x\mathrm{D}_x)(v)[\mathrm{L}_x^{-1}\theta\eta_x + c]\rangle| &\lesssim \|\eta_x\| + \|\mathrm{L}_x^{-1}\theta(\mathrm{d}_x\mathrm{D}_x)(v)\mathrm{L}_x^{-1}\theta\eta_x\| \\ &\lesssim \|\eta_x\| + \|(\mathrm{d}_x\mathrm{D}_x)(v)\mathrm{L}_x^{-1}\theta\eta_x\|, \end{aligned}$$

again by Lemma (4.16) where the implicit constant is independent of x since $(\mathrm{d}_x\mathrm{D}_x)(v)c_x = 0$ by the fact that $c_x \in \mathcal{N}(\nabla)$. For the last term, note that

$$\|c'_x\| \lesssim \|\eta'_{x,v}\| \leq \|(\mathrm{d}_x\eta_x)(v)\| + \|(\mathrm{d}_x\mathrm{D}_x)\mathrm{L}_x^{-1}\theta\eta_x\|.$$

By these calculations, it suffices to show that $\|(\mathrm{d}_x\mathrm{D}_x)(v)\mathrm{L}_x^{-1}\theta\eta_x\| \lesssim \|\eta_x\|$, where the implicit constant depends on Ω. In order to estimate this term, note that by by Proposition 4.12 $\|(\mathrm{d}_x\mathrm{D}_x)(v)u\| \lesssim \|\Delta_{\mathrm{g}}u\| + \|u\|$ uniformly in $x \in \Omega$, and therefore,

$$\begin{aligned} \|(\mathrm{d}_x\mathrm{D}_x)(v)\mathrm{L}_x^{-1}\theta\eta_x\| &\lesssim \|\Delta_{\mathrm{g}}\mathrm{L}_x^{-1}\theta\eta_x\| + \|\mathrm{L}_x^{-1}\theta\eta_x\| \\ &\lesssim \|\eta_x\| + \|\mathrm{L}_x^{-1}\theta\eta_x\| + \|\mathrm{L}_x^{-1}\theta\eta_x\| \lesssim \|\eta_x\|. \end{aligned}$$

To prove higher differentiability and continuity for $x \in \mathcal{N}$, it suffices to repeat the argument upon replacing $\eta'_{x,v}$ and ω_x, *mutatis mutandis*, to solve for higher weak derivatives. It is easy to see that this procedure can only be repeated as many times as the minimum of the regularity of η_x and ω_x. $\square$

5. The flow for rough metrics with Lipschitz kernels

From this section onward, we assume that the heat kernel improves to be differentiable on the non-empty open region $\mathcal{N}$, i.e., we assume that for some $k \geq 1$, $\rho^{g}_t \in C^{k,\alpha}(\mathcal{N}^2)$. Moreover, for the purpose of our analysis, we assume that $\rho^{g}_t \in C^{0,1}(\mathcal{M}^2)$, i.e., that it is a Lipschitz function on $\mathcal{M}^2$. The set $\mathcal{N}$ will typically be such that $\mathcal{M} \setminus \mathcal{N}$ is a set of null measure.

Now, fix $t > 0$, $x \in \mathcal{N}$, and $v \in T_x\mathcal{M}$, and recall the following linear PDE satisfying, for $\varphi_{t,x,v} \in W^{1,2}(\mathcal{M})$,

$$\text{(CE)} \qquad \begin{aligned} &-\operatorname{div}_g(\rho^{g}_t(x,y)\nabla\varphi_{t,x,v}(y)) = (d_x\rho^{g}_t(x,y))(v) \\ &\int_{\mathcal{M}} \varphi_{t,x,v}(y)\ d\mu_g(y) = 0. \end{aligned}$$

The flow of Gigli-Mantegazza defined in [13], is then given by

$$\text{(GM)} \qquad g_t(u,v)(x) = \int_{\mathcal{M}} g(\nabla\varphi_{t,x,u}(y), \nabla\varphi_{t,x,v}(y))\ \rho^{g}_t(x,y)\ d\mu_g(y).$$

5.1. **Heat kernels for the rough metric Laplacian.** The Laplacian for a rough metric is the non-negative self-adjoint operator $\Delta_g = -\operatorname{div}_g \nabla$, the operator associated with the real-symmetric form $J[u,v] = \langle\nabla u, \nabla v\rangle_g$.

Recall, we say that $\rho_g : \mathbb{R}_+ \times \mathcal{M} \times \mathcal{M}$ is the heat kernel of Δ_g if it is the minimal solution $\rho^{g}_t : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ to the heat equation

$$\text{(HK)} \qquad \begin{aligned} &\partial_t\rho^{g}_t(x,) = \Delta_0\rho^{g}_t(x,\cdot) \\ &\lim_{t\to 0}\partial_t\rho^{g}_t(x,\cdot) = \delta_x, \end{aligned}$$

where δ_x is the Dirac mass at $x \in \mathcal{M}$, satisfying

$$\rho^{g}_t(x,y) = \rho^{g}_t(y,x), \quad \rho^{g}_t(x,y) \geq 0, \quad \text{and} \quad \int_{\mathcal{M}} g_t(x,y)\ d\mu_g(y) = 1.$$

Given an initial $u_0 \in L^2(\mathcal{M})$, we are able to write

$$e^{-t\Delta_g}u_0(x) = \int_{\mathcal{M}} \rho^{g}_t(x,y)u(y)\ d\mu_g(y)$$

for almost-every $x \in \mathcal{M}$.

The following guarantees the existence of a heat kernel and its regularity properties.

Theorem 5.1. *The heat kernel for Δ_g exists and there exists $\alpha > 0$ such that $\rho^{g}_t \in C^{\alpha}(\mathcal{M}^2)$.*

Proof. This can be obtained as a direct consequence of Theorem 7.4 in [26] by Sturm, where $(X,d,m) = (\mathcal{M}, d_{\tilde{g}}, \theta^{-1}\mu_{\tilde{g}})$, where $\mu_g = \theta^{-1}\mu_{\tilde{g}}$, and where the Dirichlet energy $E(f) = \langle\nabla f, \nabla f\rangle_g$ with $\mathcal{D}(E) = W^{1,2}(\mathcal{M})$. Indeed, the measure μ_g is Radon as asserted by Proposition 2.3. Moreover, the *measure contraction property*: for any compact set $Y \subset \mathcal{M}$, there exists $\Phi_t(x, A) : [0,1] \to \mathcal{M}$ and a constant $C' > 0$ such that such that

$$\frac{\mu_{\tilde{g}}(A)}{\mu_{\tilde{g}}(B_r(x))} \leq C' \frac{\mu_{\tilde{g}}(\Phi_t(x,A))}{\mu_{\tilde{g}}(B_{tr}(x))}$$

is valid for $(\mathcal{M}, d_{\tilde{g}}, \tilde{g})$ by the virtue of the fact this space is a smooth Riemannian manifold (see Example 1D in [26]). But since the aforementioned Proposition gives that $C^{-n/2}\mu_{g} \leq \mu_{\tilde{g}} \leq C^{n/2}\mu_{\tilde{g}}$, we have that the mean contraction property holds holds for the measure μ_{g}. The fact that we obtain a C^α regular heat kernel for $\alpha > 0$ is because we are in a compact space, and local Hölder regularity improves to Hölder regularity automatically. □

We remark that this result can be obtained directly by considering uniformly elliptic operators with L^∞ coefficients via the results of Saloff-Coste in §6 in his paper [25].

Before we present the main theorem regarding the existence of this metric when the initial metric is rough, we present the following two lemmas which allow us to assert the non-degeneracy of this metric, when it exists.

Lemma 5.2 (Backward uniqueness of the heat flow)**.** *Let $u_t \in \mathrm{L}^2(\mathcal{M})$ be a strict solution to the heat equation $\partial_t u_t = \Delta_{g} u_t$ with $\lim_{t\to 0} u(t,x) = \Xi$, where $\Xi \in \mathrm{W}^{-1,2}(\mathcal{M})$ is a distribution. If there exists some $t_0 > 0$ such that $u_{t_0} = 0$, then $u_t = 0$ for all $t > 0$ and $\lim_{t\to 0} u(t,\cdot) = 0$ in the sense of distributions.*

Proof. First, suppose that $\Xi = v \in \mathrm{L}^2(\mathcal{M})$. Then, $u(t,x) = \mathrm{e}^{-t\Delta_{g}} v$ and we note that

$$\langle v, \mathrm{e}^{-t\Delta_{g}} v\rangle_{g} = \langle v, \mathrm{e}^{-\frac{1}{2}t\Delta_{g}}\mathrm{e}^{-\frac{1}{2}t\Delta_{g}} v\rangle_{g} = \langle \mathrm{e}^{-\frac{1}{2}t\Delta_{g}} v, \mathrm{e}^{-\frac{1}{2}t\Delta_{g}} v\rangle_{g} = \|\mathrm{e}^{-\frac{1}{2}t\Delta_{g}} v\|,$$

where the second equality follows by the self-adjointness of Δ_{g}. Thus, at $t = t_0$, we obtain that $\|\mathrm{e}^{-\frac{1}{2}t_0\Delta_{g}} v\| = 0$ and by induction, $\mathrm{e}^{\frac{1}{2^n}t_0\Delta_{g} v} = 0$. Hence,

$$v = \lim_{t\to 0} \mathrm{e}^{-t\Delta_{g}} v = \lim_{n\to\infty} \mathrm{e}^{\frac{1}{2^n}t_0\Delta_{g} v} = 0.$$

Now, for the case of an arbitrary distribution Ξ, we note that for $s > 0$, $u_{t+s} = \mathrm{e}^{-t\Delta_{g}} u_s$ and therefore, applying our previous argument with $v = u_s$, we obtain that $u_s = 0$ for every $s > 0$. Now, fix $f \in \mathrm{C}^\infty_{c}(\mathcal{M})$, a test function, and since $\langle \cdot, \cdot \rangle$ extends continuously to a pairing $\mathrm{W}^{-1,2}(\mathcal{M}) \times \mathrm{C}^\infty_{c}(\mathcal{M})$, we obtain that

$$0 = \lim_{t\to 0} \int_{\mathcal{M}} u_t(x) f(x)\ d\mu_{g}(x).$$

That is, $\Xi = 0$. □

Also, we have the following.

Lemma 5.3. *The function $y \mapsto \rho^{g}_t(x,y) \in \mathcal{D}(\Delta_{g})$ and for all $t > 0$, $\partial_t \rho^{g}_t(x,\cdot) = \Delta_{g}\rho^{g}_t(x,\cdot)$ for each $x \in \mathcal{M}$. If $\varnothing \neq \mathcal{N}$ is an open subset on which $(x,y) \mapsto \rho^{g}_t \in \mathrm{C}^k(\mathcal{N}^2)$ (for $k \geq 1$), then for every $x \in \mathcal{N}$ and $v \in \mathrm{T}_x\mathcal{M}$, $y \mapsto (\mathrm{d}_x \rho^{g}_t(x,y))(v)$ solves*

$$\partial_t(\mathrm{d}_x\rho^{g}_t(x,\cdot))(v) = \Delta_{g}(\mathrm{d}_x\rho^{g}_t(x,\cdot))(v), \qquad \lim_{t\to 0}(\mathrm{d}_x\rho^{g}_t(x,\cdot))(v) = D_{x,v},$$

where $D_{x,v} \in \mathrm{W}^{-1,2}(\mathcal{M})$ is given by $D_{x,v} f = (\mathrm{d}_x f)(v)$.

Proof. The fact that $x \mapsto \rho^{\mathrm{g}}_t(x,y) \in \mathcal{D}(\Delta_{\mathrm{g}})$ and $\partial_t \rho^{\mathrm{g}}_t(x,\cdot) = \Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(x,\cdot)$ for $t > 0$ is by definition that $\rho^{\mathrm{g}}_t : \mathcal{M}^2 \to \mathbb{R}_+$ is the fundamental solution to the heat equation.

First, note that

$$\partial_t(\mathrm{d}_x \rho^{\mathrm{g}}_t(x,\cdot))(v) = \mathrm{d}_x(\partial_t \rho^{\mathrm{g}}_t(x,\cdot))(v) = \mathrm{d}_x(\Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(x,\cdot))(v).$$

Now, fix $u \in \mathcal{D}(\Delta_{\mathrm{g}})$ and, fix a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathcal{N}$ such that $\gamma(0) = x$, $\dot\gamma(0) = v$, and observe that

$$\begin{aligned}
\langle (\mathrm{d}_x \Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(x,\cdot))(v), u\rangle_{\mathrm{g}} &= \int_{\mathcal{M}} (\mathrm{d}_x \Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(x,y))(v) u(y)\ d\mu_{\mathrm{g}}(y) \\
&= \int_{\mathcal{M}} \frac{d}{ds}|_{s=0} \Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(\gamma(s),y) u(y)\ d\mu_{\mathrm{g}}(y) \\
&= \frac{d}{ds}|_{s=0} \int_{\mathcal{M}} \Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(\gamma(s),y) u(y)\ d\mu_{\mathrm{g}}(y) \\
&= \frac{d}{ds}|_{s=0} \int_{\mathcal{M}} \rho^{\mathrm{g}}_t(\gamma(s),y) \Delta_{\mathrm{g}} u(y)\ d\mu_{\mathrm{g}}(y) \\
&= \int_{\mathcal{M}} \frac{d}{ds}|_{s=0} \rho^{\mathrm{g}}_t(\gamma(s),y) \Delta_{\mathrm{g}} u(y)\ d\mu_{\mathrm{g}}(y) \\
&= \langle (\mathrm{d}_x \rho^{\mathrm{g}}_t(x,\cdot))(v), \Delta_{\mathrm{g}} u\rangle_{\mathrm{g}}.
\end{aligned}$$

This shows that $u \mapsto \langle (\mathrm{d}_x \rho^{\mathrm{g}}_t(x,\cdot))(v), \Delta_{\mathrm{g}} u\rangle = \langle (\mathrm{d}_x \Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(x,\cdot))(v), u\rangle_{\mathrm{g}}$ is continuous in u and hence $(\mathrm{d}_x \rho^{\mathrm{g}}_t(x,\cdot))(v) \in \mathcal{D}(\Delta_{\mathrm{g}})$ and by a similar calculation,

$$\langle \partial_t(\mathrm{d}_x \Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(x,\cdot))(v), u\rangle_{\mathrm{g}} = \langle \Delta_{\mathrm{g}}(\mathrm{d}_x \rho^{\mathrm{g}}_t(x,\cdot))(v), u\rangle_{\mathrm{g}}.$$

Since $\mathcal{D}(\Delta_{\mathrm{g}})$ is dense in $\mathrm{L}^2(\mathcal{M})$, we obtain that $(\mathrm{d}_x \Delta_{\mathrm{g}} \rho^{\mathrm{g}}_t(x,\cdot))(v)$ solves the heat equation.

Now, fix $f \in \mathrm{C}^\infty_{\mathrm{c}}(\mathcal{M})$. Then,

$$\begin{aligned}
\lim_{t\to 0} \int_{\mathcal{M}} (\mathrm{d}_x \rho^{\mathrm{g}}_t(x,y))(v) f(y)\ d\mu_{\mathrm{g}}(y) &= \lim_{t\to 0} \int_{\mathcal{M}} \frac{d}{ds}|_{s=0} \rho^{\mathrm{g}}_t(\gamma(s),y) f(y)\ d\mu_{\mathrm{g}}(y) \\
&= \lim_{t\to 0} \frac{d}{ds}|_{s=0} \int_{\mathcal{M}} \rho^{\mathrm{g}}_t(\gamma(s),y) f(y)\ d\mu_{\mathrm{g}}(y) \\
&= \frac{d}{ds}|_{s=0} \lim_{t\to 0} \int_{\mathcal{M}} \rho^{\mathrm{g}}_t(\gamma(s),y) f(y)\ d\mu_{\mathrm{g}}(y) \\
&= \frac{d}{ds}|_{s=0} f(\gamma(s)) \\
&= (\mathrm{d}_x f)(v).
\end{aligned}$$

□

In order to apply the elliptic tools we've described in the previous sections, we need to assert that $\rho^{\mathrm{g}}_t(x,y) > 0$. This is the content of the following lemma.

Lemma 5.4. *For each $t > 0$, there exist $0 < \kappa_t, \Lambda_t < \infty$ such that $\kappa_t \le \rho^{\mathrm{g}}_t(x,y) \le \Lambda_t$.*

Proof. Fix $z \in \mathcal{M}$ and note that $y \mapsto \rho^{\mathrm{g}}_t(z,y)$ is again a solution to the heat equation. Now, assume that at some time t_0, $\rho^{\mathrm{g}}_{t_0}(z,x) = 0$. Then,

$$0 = \int_{\mathcal{M}} \rho^{\mathrm{g}}_{\frac{1}{2}t_0}(x,y)\rho^{\mathrm{g}}_{\frac{1}{2}t_0}(z,y)\ d\mu_{\mathrm{g}}(y).$$

Since $\rho^{\mathrm{g}}_t(x,y) \geq 0$ and is at least continuous, we have that either $\rho^{\mathrm{g}}_{\frac{1}{2}t_0}(x,y) = 0$ or $\rho^{\mathrm{g}}_{\frac{1}{2}t_0}(z,y) = 0$ for all $y \in \mathcal{M}$.

Now, by Lemma 5.2, we can obtain that either $0 = \lim_{t\to 0} \rho^{\mathrm{g}}_t(x,\cdot\,) = \delta_x$ or $0 = \lim_{t\to 0} \rho^{\mathrm{g}}_t(z,\cdot\,) = \delta_z$, both of which cause a contradiction. This shows that $\rho^{\mathrm{g}}(x,y) > 0$ for each $t > 0$ and for all $x, y \in \mathcal{M}$.

By the compactness of $\mathcal{M}$, we obtain that

$$\begin{aligned} \kappa_1 &= \inf_{x,y\in\mathcal{M}} \rho^{\mathrm{g}}_t(x,y) = \min_{x,y\in\mathcal{M}} \rho^{\mathrm{g}}_t(x,y) > 0, \text{ and} \\ \Lambda_1 &= \sup_{x,y\in\mathcal{M}} \rho^{\mathrm{g}}_t(x,y) = \max_{x,y\in\mathcal{M}} \rho^{\mathrm{g}}_t(x,y) < \infty. \end{aligned}$$

This completes the proof. □

5.2. **The flow.** We collate the results we have obtained so far and present the following existence and regularity theorem for g_t.

Proof of Theorem 3.1. First, we show that for each $t > 0$, $x \in \mathcal{M}$ and $v \in \mathrm{T}_x\mathcal{M}$, there exists a unique $\varphi_{t,x,v} \in \mathrm{W}^{1,2}(\mathcal{M})$ which solves (CE). By Lemma 5.4, we are able to apply Proposition 4.7 on setting $\omega_x(y) = \omega(x,y) = \rho^{\mathrm{g}}_t(x,y)$ and $\eta(y) = \mathrm{d}_x(\rho^{\mathrm{g}}_t(x,y))(v)$.

The only thing that needs to be to be checked is that $\int_{\mathcal{M}} \eta\ d\mu_{\mathrm{g}} = 0$. In order to do so, let $\gamma : I \to \mathcal{M}$ be a curve so that $\gamma(0) = x$ and $\dot\gamma(0) = v$. Then, on noting that

$$(\mathrm{d}_x(\rho^{\mathrm{g}}_t(x,y)))(v) = \frac{d}{ds}|_{s=0}\rho^{\mathrm{g}}_t(\gamma(s),y),$$

we compute

$$\begin{aligned} \int_{\mathcal{M}} \frac{d}{ds}|_{s=0}\rho^{\mathrm{g}}_t(\gamma(s),y)\ d\mu_{\mathrm{g}} &= \int_{\mathcal{M}} \frac{d}{ds}|_{s=0}\rho^{\mathrm{g}}_t(\gamma(s),y)\ d\mu_{\mathrm{g}} \\ &= \frac{d}{ds}|_{s=0}\int_{\mathcal{M}} \rho^{\mathrm{g}}_t(\gamma(s),y)\ d\mu_{\mathrm{g}} = \frac{d}{ds}|_{s=0}1 = 0, \end{aligned}$$

where in the second line, we have used the dominated convergence theorem to interchange the integral and the limit involved in differentiation. Thus, on invoking Proposition 4.7, we obtain a unique solution $\varphi_{t,x,v} \in \mathrm{W}^{1,2}(\mathcal{M})$ with $\int_{\mathcal{M}} \varphi_{t,x,v}\ d\mu_{\mathrm{g}} = 0$. It is easy to see then that g_t is symmetric at each x.

Next, we show that $\nabla\varphi_{t,x,v} = 0$ if and only if $v = 0$. Fix $v \neq 0$, and recall Lemmas 5.2 and 5.3 to conclude that $(\mathrm{d}_x\rho^{\mathrm{g}}_t(x,\cdot\,))(v) \neq 0$ since $\lim_{t\to 0}(\mathrm{d}_x\rho^{\mathrm{g}}_t(x,\cdot\,))(v) = D_{x,v} \neq 0$. Since the solution provided by Proposition 4.5 is obtained by inverting the one-one operator L^R_A in Proposition 4.5, we must have that $\psi_{t,x,v} \notin \mathcal{N}(\mathrm{L}^R_A) = \mathcal{N}(\nabla)$. It is easy to see that if $v = 0$, then $(\mathrm{d}_x \rho^{\mathrm{g}}_t(x,\cdot))(v) = 0$ and hence, $\varphi_{t,x,v} = 0$. This shows that $\mathrm{g}_t(u,u)(x) \geq 0$ and $\mathrm{g}_t(u,u)(x) = 0$ if and only if $u = 0$. That is, g_t is non-degenerate.

Furthermore, it is easy to see that, for $\alpha \neq 0$, $\alpha\varphi_{t,x,v}$ solves the equation (CE) with source term $(\mathrm{d}_x \rho^{\mathrm{g}}_t(x,\cdot))(\alpha v)$ and that, by linearity of the equation (CE), $\varphi_{t,x,v} + \varphi_{t,x,u}$ solves (CE) with source term $(\mathrm{d}_x \rho^{\mathrm{g}}_t(x,\cdot))(u+v)$. Hence, $\alpha\varphi_{t,x,v} = \varphi_{t,x,\alpha v}$ and $\varphi_{t,x,v} + \varphi_{t,x,u} = \varphi_{t,x,u+v}$. That is, $\mathrm{g}_t(\alpha u, v)(x) = \alpha \mathrm{g}_t(u,v)(x)$ and $\mathrm{g}_t(u+v,w)(x) = \mathrm{g}_t(u,w)(x) + \mathrm{g}_t(v,w)(x)$. Thus, g_t is linear in the first variable.

By combining symmetry, non-degeneracy, and linearity in the first variable shows that $\mathrm{g}_t(x) : \mathrm{T}_x\mathcal{M} \to \mathrm{T}_x\mathcal{M} \to \mathbb{R}_{>0}$ defines an inner product on $\mathrm{T}_x\mathcal{M}$ and hence, a Riemannian metric.

Regularity is then a simple consequence of Theorem 4.17 since $|u|^2_{\mathrm{g}_t(x)} = \langle \eta_{x,u}, \varphi_{x,u}\rangle$, and the same regularity can be obtained for $x \mapsto \mathrm{g}_t(u,v)(x)$ via polarisation. ☐

6. Regularity of the flow for sufficiently smooth metrics

In [13], the authors demonstrate that the flow g_t is smooth for all positive times when starting with a smooth initial metric. We demonstrate this similar result but when the metric is assumed to be $\mathrm{C}^{k,\alpha}$, where $k \geq 1$. Our approach is to demonstrate that we are able to localise our weak solutions, and then, for this more regular class of metrics, apply Schauder theory to obtain higher $(k+1)$ regularity for the heat kernel ρ^{g}_t. On applying Theorem 4.17, we are able to assert that g_t remains C^k.

6.1. **Higher regularity of the flow for C^1 heat kernels.**

First, we demonstrate that for a C^1 heat kernel, the regularity theorem (see Theorem 4.17) improves from $\mathrm{C}^{k-1,1}$ to C^k.

Recall that $\mathrm{L}_x u = \mathrm{div}_{\tilde{\mathrm{g}}}(A\theta\omega_x)\nabla u$. We estimate the difference between such operators. We fix $f : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$, with $-\Lambda \leq f(x,y) \leq \Lambda$ for $x \in U$, where U is an open set. Define $\Xi : U \times U \to \mathbb{R}_{\geq 0}$ by

$$\Xi(x,y) = \|f_x - f_y\|_\infty + \|\nabla(f_x - f_y)\|_\infty,$$

where $f_x = f(x,\cdot)$.

Lemma 6.1. *Let $(x,y) \mapsto f_x \in \mathrm{C}^1(\mathcal{M}^2)$ and let $\Lambda > 0$ such that $-\Lambda \leq f_x(y) \leq \Lambda$ for $x \in U$, where U is an open set and all $y \in \mathcal{M}$. Define $T_x u = -\mathrm{div}_{\mathrm{g}} f_x \nabla u$ with domain $\mathcal{D}(T_x) = \mathcal{D}(\Delta_{\mathrm{g}})$. Then, whenever $u \in \mathcal{D}(\Delta_{\mathrm{g}})$,*

$$\|T_x u - T_y u\| \lesssim \Xi(x,y)\|u\|_{\Delta_{\mathrm{g}}}.$$

whenever $x, y \in U$ and where the implicit constant depends on U.

Proof. Define $F_x = f_x + 2\Lambda$ and it follows that $\Lambda \leq F_x \leq 3\Lambda$ on $U \times \mathcal{M}$. On setting $S_x u = -\mathrm{div}_{\mathrm{g}} F_x \nabla u$, by Proposition 4.8, we obtain that $\mathcal{D}(S_x) = \mathcal{D}(\Delta_{\mathrm{g}})$ and

$S_x u = F_x \Delta_{\mathrm{g}} u + \mathrm{g}(\nabla u, \nabla F_x)$. It is easy to check that $T_x u = S_x u - 2\Lambda \Delta_{\mathrm{g}} u$ and therefore, $T_x u - T_y u = S_x u - S_y u$.

We compute,

$$\|S_x u - S_y u\| \leq \|(F_x - F_y)\Delta_{\mathrm{g}} u\| + \|\mathrm{g}(\nabla u, \nabla (F_x - F_y)\|,$$

but $F_x - F_y = f_x - f_y$, and hence, it follows that

$$\|T_x u - T_y u\| \leq \|f_x - f_y\|_\infty \|\Delta_{\mathrm{g}} u\| + \|\nabla (f_x - f_y)\|_\infty \|\nabla u\|.$$

The estimate $\|\nabla u\| \lesssim \|u\|_{\Delta_{\mathrm{g}}}$ is trivial, and so the claim is proved. □

We have a similar result for the resolvents L_x^{-1} on the range of the operator L_x.

Lemma 6.2. *Suppose that $(x, y) \mapsto \omega_x(y) \in \mathrm{C}^1(\mathcal{M}^2)$ and let $u_1, u_2 \in \mathrm{L}^2(\mathcal{M})$ satisfy $\int_{\mathcal{M}} u_1 \, d\mu_{\tilde{\mathrm{g}}} = \int_{\mathcal{M}} u_2 \, d\mu_{\tilde{\mathrm{g}}} = 0$. Then,*

$$\|\mathrm{L}_x^{-1} u_1 - \mathrm{L}_y^{-1} u_2\| \lesssim \Xi(x, y)\|u_1\| + \|u_1 - u_2\|.$$

The implicit constant is independent of x and this expression is valid for all $x, y \in \mathcal{M}$.

Proof. First note that for $v \in \mathcal{D}(\Delta_{\mathrm{g}}) = \mathcal{D}(\mathrm{L}_x)$,

$$\|\mathrm{L}_x v - \mathrm{L}_y v\| = \|\theta(\mathrm{D}_x v - \mathrm{D}_y v)\| \lesssim \|\mathrm{D}_x v - \mathrm{D}_y v\| \lesssim \Xi(x, y)\|v\|_{\Delta_{\mathrm{g}}}$$

by invoking Lemma 6.1 with $U = \mathcal{M}$.

Now, fix $u \in \mathrm{L}^2(\mathcal{M})$ with $\int_{\mathcal{M}} u \, d\mu_{\tilde{\mathrm{g}}} = 0$ and note that $\mathrm{L}_x^{-1} u = \mathrm{L}_x^{-1}(\mathrm{L}_y \mathrm{L}_y^{-1})u = (\mathrm{L}_x^{-1}\mathrm{L}_y)\mathrm{L}_y^{-1})u$. Also, $\mathrm{L}_y^{-1} u = \mathrm{L}_x^{-1} \mathrm{L}_x L_y^{-1} u$ since the resolvent and operator commute on its domain. Thus,

$$\begin{aligned}\|\mathrm{L}_x^{-1} u - \mathrm{L}_y^{-1} u\| = \|\mathrm{L}_x^{-1}\mathrm{L}_y \mathrm{L}_y^{-1} u - \mathrm{L}_x^{-1}\mathrm{L}_x \mathrm{L}_y^{-1} u\| &= \|\mathrm{L}_x^{-1}(\mathrm{L}_y - \mathrm{L}_x)\mathrm{L}_y^{-1} u\| \\ &\lesssim \|(\mathrm{L}_y - \mathrm{L}_x)\mathrm{L}_y^{-1} u\| \lesssim \Xi(x, y)\|\mathrm{L}_y^{-1} u\|_{\Delta_{\mathrm{g}}},\end{aligned}$$

since by Lemma 4.16, we have that $\|\mathrm{L}_x^{-1} u\| \lesssim \|u\|$ independent of x. By Proposition 4.10, we obtain that $\|v\|_{\Delta_{\mathrm{g}}} \simeq \|v\|_{\mathrm{D}_y}$, independent of y and that $\|v\|_{\mathrm{D}_y} \simeq \|v\|_{\mathrm{L}_y}$. Hence, on setting $v = \mathrm{L}_y^{-1} u$, we obtain that $\|\mathrm{L}_y^{-1} u\|_{\Delta_{\mathrm{g}}} \lesssim \|u\|$.

Now, for u_1 and u_2 as in the hypothesis,

$$\begin{aligned}\|\mathrm{L}_x^{-1} u_1 - \mathrm{L}_y^{-1} u_2\| \leq \|\mathrm{L}_x^{-1} u_1 - \mathrm{L}_y^{-1} u_1\| + \|\mathrm{L}_y^{-1}(u_1 - u_2)\| \\ \lesssim \Xi(x, y)\|u_1\| + \|u_1 - u_2\|.\end{aligned}$$

□

With the aid of these two lemmas, we improve the regularity from Theorem 4.17 as follows.

Theorem 6.3. *Suppose that $(x, y) \mapsto \omega_x(y) \in \mathrm{C}^1(\mathcal{M}^2)$ and that $x \mapsto \omega_x \in \mathrm{C}^k(\mathcal{N})$ for $k \geq 1$. Moreover, suppose that $(x, y) \mapsto \eta_x(y) \in \mathrm{C}^0(\mathcal{N} \times \mathcal{M})$ and that $x \mapsto \eta_x \in \mathrm{C}^l(\mathcal{N})$ for $l \geq 0$. If at $x \in \mathcal{N}$, φ_x solves (F) with $\int_{\mathcal{M}} \varphi_x \, d\mu_{\mathrm{g}} = \int_{\mathcal{M}} \eta \, d\mu_{\mathrm{g}} = 0$. Then, $x \mapsto \langle \eta_x, \varphi_x \rangle \in \mathrm{C}^{\min\{k,l\}}(\mathcal{M})$.*

Proof. First, suppose that $l = 0$. Then, we show that $x \mapsto \langle \eta_x, \varphi_x \rangle \in \mathrm{C}^0(\mathcal{N})$. For that, note that

$$|\langle \eta_x, \varphi_x \rangle - \langle \eta_y, \varphi_y \rangle| \leq |\langle \eta_x - \eta_y, \varphi_x \rangle| + |\langle \eta_y, \varphi_x - \varphi_y \rangle| \leq \|\eta_x - \eta_y\| \|\varphi_x\| + \|\eta_y\| \|\varphi_x - \varphi_y\|.$$

Since we assume that $(x, z) \mapsto \eta_x(z) \in \mathrm{C}^0(\mathcal{N} \times \mathcal{M})$, the same is true for $(x, z) \in \overline{U} \times \mathcal{M}$, where $\overline{U} \subset \mathcal{N}$ is a compact set with nonempty interior containing x and hence, $\langle \eta_x - \eta_y \rangle$ can be made small. Now, the term $\varphi_x = \mathrm{L}_x^{-1}\theta\eta_x - \fint_{\mathcal{M}} \mathrm{L}_x^{-1}\theta\eta_x \, d\mu_{\mathrm{g}}$ and hence, it suffices to show that $\|\mathrm{L}_x^{-1}\theta\eta_x - \mathrm{L}_y^{-1}\theta\eta_y\|$ can be made small. For this, note that

$$\|\mathrm{L}_x^{-1}\theta\eta_x - \mathrm{L}_y^{-1}\theta\eta_y\| \lesssim \Xi(x, y)\|\eta_x\| + \|\eta_x - \eta_y\|$$

by Lemma 6.2, and hence, this term can also be made small when y is sufficiently close to x.

Next, we note that by bootstrapping, it suffices to consider the situation where $k, l = 1$ and we note that Theorem 4.17 gives us that $x \mapsto \langle \eta_x, \varphi_x \rangle$ has a bounded derivative. All we need to prove is that this derivative is continuous.

We recall that, via the product rule for the weak derivative, we write inside a chart,

$$(\partial_i \langle \eta_x, \varphi_x \rangle)(v) = \langle (\partial_i \eta_x, \varphi_x \rangle + \langle \eta_x, \partial_i \varphi_x \rangle,$$

and hence, we show that each term of the right hand side is continuous.

Fix $x, y \in U \subset \mathcal{M}$ open , where $(\psi, \tilde{U})$ is a chart with $\overline{U} \subset \tilde{U}$ compact. Then, we have

$$\langle \partial_i \eta_x, \varphi_x \rangle - \langle \partial_i \eta_y, \varphi_y \rangle \leq \langle \partial_i \eta_x - \partial_i \eta_y, \varphi_x \rangle + \langle \partial_i \eta_y, \varphi_x - \varphi_y \rangle.$$

Now, since $x \mapsto \eta_x$ is C^1 by assumption, $\|\partial_i \eta_x - \partial_i \eta_y\|$ can be made small.

In the continuity case, we have already shown that $\|\varphi_x - \varphi_y\|$ can be made small, so we consider the next term

$$\langle \eta_x, \partial_i \varphi_x \rangle - \langle \eta_y, \partial_i \varphi_y \rangle = \langle \eta_x - \eta_y, \partial_i \varphi_x \rangle + \langle \eta_y, \partial_i \varphi_x - \partial_i \varphi_y \rangle.$$

Now, it is easy to see that the first term on the right hand side is trivially continuous because $\|\eta_x - \eta_y\|$ can be made small. The continuity for the second term follows by showing that $\|\partial_i \varphi_x - \partial_i \varphi_y\|$ can be made small. Recall that $\partial_i \varphi_x = \mathrm{L}_x^{-1}\theta\eta'_{x,i} - \fint_{\mathcal{M}} \mathrm{L}_x^{-1}\theta\eta'_{x,i}$, where $\eta'_{x,i} = \partial_i \eta_x - (\partial_i \mathrm{D}_x)\varphi_x$. Thus, it suffices to prove that $\|\mathrm{L}_x^{-1}\theta\eta'_{x,i} - \mathrm{L}_y^{-1}\theta\eta'_{y,i}\|$ can be made small. By Lemma 6.2, we have that

$$\|\mathrm{L}_x^{-1}\theta\eta_{x,i} - \mathrm{L}_y^{-1}\theta\eta_{y,i}\| \lesssim \Xi(x, y)\|\eta_{x,i}\| + \|\mathrm{L}_y^{-1}\theta(\eta_{x,i} - \eta_{y,i})\|.$$

Hence, we are reduced to proving that $\|\eta'_{x,i} - \eta'_{y,i}\|$ can be made small.

Now, note that $(\partial_i \mathrm{D}_x)\varphi_x = (\partial_i \mathrm{D}_x)\mathrm{L}_x^{-1}\theta\eta_x$ since $(\partial_i \mathrm{D}_x)(\fint_{\mathcal{M}} \mathrm{L}_x^{-1}\theta\eta_x \, d\mu_{\mathrm{g}}) = 0$ and thus,

$$\|\eta'_{x,i} - \eta'_{y,i}\| \leq \|\partial_i \eta_x - \partial_i \eta_y\| + \|(\partial_i \mathrm{D}_x)\mathrm{L}_x^{-1}\theta\eta_x + (\partial_i \mathrm{D}_y)\mathrm{L}_y^{-1}\theta\eta_y\|.$$

It is easy to see that the first term can be made small, so we only need to show that the second term can be made small. Now,

$$\|(\partial_i \mathrm{D}_x)\mathrm{L}_x^{-1}\theta\eta_x - (\partial_i \mathrm{D}_y)\mathrm{L}_y^{-1}\theta\eta_y\| \leq \|[(\partial_i \mathrm{D}_x) - (\partial_i \mathrm{D}_y)]\mathrm{L}_x^{-1}\theta\eta_x\| + \|(\partial_i \mathrm{D}_y)(\mathrm{L}_x^{-1}\theta\eta_x - \mathrm{L}_y^{-1}\theta\eta_y)\|,$$

and by Lemma 6.1, we have that

$$\|[(\partial_i \mathrm{D}_x) - (\partial_i \mathrm{D}_y)]\mathrm{L}_x^{-1}\theta\eta_x\| \lesssim \Xi(x,y)\|\mathrm{L}_x^{-1}\theta\eta_x\|_{\Delta_{\mathrm{g}}} \lesssim \Xi(x,y)\|\eta_x\|.$$

For the remaining term, as in the proof of Lemma 6.2, we write,

$$\mathrm{L}_x^{-1}\theta\eta_x - \mathrm{L}_y^{-1}\theta\eta_y = \mathrm{L}_x^{-1}(\mathrm{L}_x - \mathrm{L}_y)\mathrm{L}_y^{-1}\theta\eta_x + \mathrm{L}_y^{-1}\theta(\eta_x - \eta_y),$$

and since $\|(\partial_i \mathrm{D}_y)\mathrm{L}_x^{-1}\| \lesssim 1$ uniformly in x and y inside U and since $\overline{U}$ is compact, we have that

$$\|(\partial_i \mathrm{D}_y)(\mathrm{L}_x^{-1}\eta_x - \mathrm{L}_y^{-1}\theta\eta_y)\| \lesssim \|(\mathrm{L}_x - \mathrm{L}_y)\mathrm{L}_y^{-1}\theta\eta_x\| + \|\mathrm{L}_y^{-1}\theta(\eta_x - \eta_y)\| \lesssim \Xi(x,y)\|\mathrm{L}_y^{-1}\theta\eta_x\|_{\Delta_{\mathrm{g}}} + \|\eta_x - \eta_y\| \lesssim \Xi(x,y)\|\eta_x\| + \|\eta_x - \eta_y\|.$$

This is again a quantity that can be made small. This shows that $x \mapsto \eta_{x,i}$ is continuous and to show that the $\min\{k,l\}$ derivative can be made continuous for $k,l \geq 1$ is obtained via a bootstrapping of this procedure. □

Remark 6.4. *Showing higher derivatives are continuous is a rather tedious task. One considers the expression solving for a second derivative (when there is sufficient regularity in $x \mapsto \omega_x$ and $x \mapsto \eta_x$) given by*

$$\mathrm{D}_x \partial_j \partial_i \varphi_x = \partial_j \partial_i \eta_x - (\partial_j \partial_i \mathrm{D}_x)\varphi_x - (\partial_i \mathrm{D}_x)\partial_j \varphi_x - (\partial_j \mathrm{D}_x)\partial_i \varphi_x.$$

The first term on the right hand side can be handled easily. The second term follows from a similar estimate as in Theorem 6.3, because $(\partial_j \partial_i \mathrm{D}_x)$ is a divergence form operator $-\operatorname{div}_{\mathrm{g}}(\partial_j \partial_i \omega_x)\nabla u$, whose coefficients satisfy $-\Lambda_U \leq \omega_x(y) \leq \Lambda_U$ for $x, y \in U$, an open neighbourhood of x for which $\overline{U} \subset \mathcal{N}$. The remaining two terms can also be handled similarly on writing $\partial_j \varphi_x$ and $\partial_i \varphi_x$ as a solution via the resolvent terms L_x^{-1} to relate back to $\eta'_{x,i}$ and to η_x.

As a corollary, we obtain an improvement of the regularity of the flow for C^1 heat kernels. Note that, unlike in Theorem 3.1, we can allow for C^1 heat kernels. This is the statement of Theorem 3.2.

6.2. **Heat kernel regularity in terms of the regularity of the metric.** In this subsection, we relate the regularity of the heat kernel to the regularity of the metric. We first prove the following important localisation lemma.

Lemma 6.5. *Suppose that $\operatorname{div}_{\mathrm{g}} A\nabla u = f$, for $u \in \mathrm{W}^{1,2}(\mathcal{M})$ and $f \in \mathrm{L}^2(\mathcal{M})$. Then, for each $x \in \mathcal{M}$, there is an $r > 0$ and a chart $\psi : U \to B_r(x')$ where $x' = \psi(x)$ and such that on $\Omega = \psi^{-1}(B_{1/2r}(x'))$,*

$$\operatorname{div}_{\tilde{\mathrm{g}},\Omega} AB\theta\nabla u = \theta f$$

in $\mathrm{L}^2(\Omega, \tilde{\mathrm{g}})$, where $\tilde{\mathrm{g}} = \psi^*\delta$, the pullback of the Euclidean metric in $B_r(x')$, $d\mu_{\mathrm{g}} = \theta\, d\tilde{\mathrm{g}}$ and $\tilde{\mathrm{g}}(Bu, v) = \mathrm{g}(u, v)$. Moreover, this equation holds if and only if

$$\operatorname{div}_{\mathbb{R}^n, B_{1/2r}(x')} \tilde{A}\tilde{B}\tilde{\theta}\nabla\tilde{u} = \tilde{\theta}\tilde{f},$$

where $\tilde{\xi} = \eta(\xi \circ \psi^{-1})$, *where* η *is a smooth cutoff which is* 1 *on* $B_{1/2r}(x)$, *and* 0 *outside* $B_{3/4r}(x)$.

Proof. Fix $v \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$. Then, for $u \in \mathcal{D}(\mathrm{div}_\mathrm{g})$, we have that

$$\langle \mathrm{div}_\mathrm{g}\, u, v\rangle = \langle u, \nabla v\rangle = \langle u, \nabla v\rangle_{\mathrm{L}^2(\Omega,\mathrm{g})} = \langle B\theta u, \nabla v\rangle_{\mathrm{L}^2(\Omega,\tilde{\mathrm{g}})}.$$

Since this holds for any such $v \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$, it follows that $B\theta u \in \mathcal{D}(\mathrm{div}_{\tilde{\mathrm{g}},\Omega})$ and hence

$$\langle B\theta u, \nabla v\rangle_{\mathrm{L}^2(\Omega,\tilde{\mathrm{g}})} = \langle \theta^{-1}\, \mathrm{div}_{\tilde{\mathrm{g}},\Omega}\, B\theta u, v\rangle_{\mathrm{L}^2(\Omega,\mathrm{g})}.$$

Thus, $\mathrm{div}_\mathrm{g}\, A\nabla u = f$ implies that $\langle \mathrm{div}_\mathrm{g}\, A\nabla u, v\rangle = \langle f, v\rangle$ for all $v \in \mathrm{C}^\infty_\mathrm{c}(\Omega)$ and hence, $\mathrm{div}_{\tilde{\mathrm{g}},\Omega}\, AB\theta\nabla u = \theta f$ in $\mathrm{L}^2(\Omega, \tilde{\mathrm{g}})$. Since η and φ induces a bijection between $\mathrm{C}^\infty_\mathrm{c}(\mathrm{B}_{1/2r})$ and $\mathrm{C}^\infty_\mathrm{c}(\Omega)$, it follows that $\mathrm{div}_{\mathbb{R}^n, B_{1/2r}(x')}\, \tilde{A}\tilde{B}\tilde{\theta}\nabla\tilde{u} = \tilde{\theta}\tilde{f}$. ☐

When the metric is sufficiently regular (i.e. at least Lipschitz), we are able to write solutions to the Laplace equation in non-divergence form.

Lemma 6.6. *Let* (ψ, U) *be a chart near* x *with* $\psi(U) = B_{1/2r}(x')$, *and suppose that* $\mathrm{g} \in \mathrm{C}^{k,\alpha}(U)$, *where* $\alpha = 1$ *if* $k = 0$ *and otherwise, for* $k \geq 1$, $\alpha \in [0, 1]$. *Then, inside* $\psi(U)$,

$$\widetilde{\Delta_\mathrm{g} u}(y) = \tilde{A}^{ij}(y)\partial_i\partial_j\tilde{u}(y) + \partial_j(\tilde{A}^{ij}\tilde{\theta})\partial_i\tilde{u},$$

for almost-every $y \in B_{1/2r}(x')$, *where* $\tilde{\xi}$ *is the notation from Lemma 6.5. The coefficients* $\tilde{A}^{ij}, \tilde{\theta}, \partial_j(\tilde{A}^{ij}) \in \mathrm{C}^{k-1,\alpha}$ *for* $k \geq 1$. *Otherwise,* $\tilde{A}^{ij}, \tilde{\theta}, \partial_j(\tilde{A}^{ij}) \in \mathrm{L}^\infty(B_{1/2r}(x'))$.

Proof. This is simply a direct consequence of Theorem 8.8 in [15]. This formula is precisely the one written in (8.18) in this theorem. ☐

Next, we obtain the first increase in regularity which allows us to initiate a bootstrapping procedure.

Lemma 6.7. *Let* (ψ, U) *be a chart near* x *and* $\psi(U) = B_r$. *Suppose that* $\mathrm{g} \in \mathrm{C}^{k,\alpha}(U)$ *for* $k \geq 1$ *and* $\alpha \in [0, 1]$ *and suppose that* $u \in \mathrm{W}^{1,2}(\mathcal{M})$ *and* $f \in \mathrm{L}^2(\mathcal{M})$ *satisfy* $\Delta_\mathrm{g} u = f$. *If* $\tilde{f} \in \mathrm{C}^\alpha(B_r(x'))$, *then* $\tilde{u} \in \mathrm{C}^{2,\alpha}(B_{1/4r}(x'))$. *Moreover,* $u|_\Omega \in \mathrm{C}^{2,\alpha}(\Omega)$ *where* $\Omega = \psi^{-1}(B_{1/4r}(x'))$.

Proof. First, set $r' = 1/2r$, and invoke the localisation from Lemma (6.5). Note that this equation $\mathrm{div}_{\mathbb{R}^n, B_{3/4r'}(x')}\, \tilde{AB\theta}\nabla\tilde{u} = \tilde{\theta f}$ is in divergence form, and since $\tilde{\theta f} \in \mathrm{C}^\alpha(\overline{B_{3/4r'}(x')})$, we have that $\tilde{\theta f} \in \mathrm{L}^q(B_{3/4r'}(x'))$ for any $q > n$. Hence, we can invoke the elliptic Harnack estimate in Theorem 8.22 in [15] to obtain that $\tilde{u} \in \mathrm{C}^\beta(B_{1/2r'}(x'))$ for some $\beta > 0$.

Next, we invoke Lemma 6.6, to write

$$\tilde{A}^{ij}(y)\partial_i\partial_j\tilde{u}(y) + \partial_j(\tilde{A}^{ij}\tilde{\theta})\partial_i\tilde{u} = \tilde{\theta}\tilde{f},$$

inside $B_{1/2r'}(x')$. Then, note that $\tilde{u}$ solves

$$L\tilde{u} = \tilde{\theta}\tilde{f}, \quad \text{with} \quad \tilde{u} = \varphi \in \mathrm{C}^0(\partial B_{3/8r'}(x')),$$

where L has $\mathrm{C}^{k-1,\alpha}$ coefficients and $\tilde{\theta}\tilde{f} \in \mathrm{C}^{\alpha}(B_{3/8r'}(x'))$ simply on setting $\varphi = \tilde{u}$ on $\partial B_{3/8r'}(x') \subset B_{1/2r'}(x')$ on which we have already proved that $\tilde{u}$ is C^{β} and hence, continuous.

Thus, we can invoke Theorem 6.13 in [15] to obtain that $\tilde{u} \in \mathrm{C}^{2,\alpha}(B_{1/2r'}(x'))$. By the definition of $\tilde{u}$, and on noting that $r' = 1/2r$, the conclusions for $u|_{\Omega}$ follow. □

With these tools in hand, we prove the following main theorem of this section. By β' we denote the a priori regularity of the heat kernel obtained from Theorem 5.1.

Theorem 6.8. *Let* $\mathrm{g} \in \mathrm{C}^{k,\alpha}(\mathcal{N})$*, where* $\varnothing \neq \mathcal{N}$ *is an open set and where* $k \geq 1$ *and* $\alpha \in [0,1]$*. Then,* $\rho^{\mathrm{g}}_t \in \mathrm{C}^{k+1,\beta}(\mathcal{N}^2)$*,where* $\beta = \min\{\alpha, \beta'\}$*.*

Proof. First, we know that the heat kernel exists and that it is at least $\mathrm{C}^{\beta'}$ for some $\beta' > 0$ by Theorem 5.1.

Fix $z \in \mathcal{M}$ and set $u(y) = \rho^{\mathrm{g}}_t(y,z)$ and $f(y) = \partial_t \rho^{\mathrm{g}}_t(y,z)$. Now, fix (ψ, U), a chart near $x \in \mathcal{N}$ so that $U \subset \mathcal{N}$ and $B_r(x') = \psi(U)$

We proceed by applying Theorem 6.17 in [15]. Define $\beta = \min\{\beta', \alpha\}$. First, let us apply the initial bootstrapping lemma, Lemma 6.7 to conclude that, in fact, $u|_{\Omega} \in \mathrm{C}^{2,\beta}(\Omega)$, where $\Omega = \psi^{-1}(B_{1/4r}(x'))$. This shows that that $u \in \mathrm{C}^{2,\beta}(\mathcal{N})$ and by the symmetry of the heat kernel, we obtain that $\rho^{\mathrm{g}}_t \in \mathrm{C}^{2,\beta}(\mathcal{N}^2)$. Thus, we have shown that the conclusion holds for $k = 1$.

Now, in the case that $k = 2$, we have that the operator L as defined in Lemma 6.7 has $\mathrm{C}^{1,\alpha}$-coefficients. Therefore, since we have that $u|_{\Omega}, f|_{\Omega} \in \mathrm{C}^{2,\beta}(\Omega)$ and, in particular, $u|_{\Omega}, f|_{\Omega} \in \mathrm{C}^{1,\beta}(\Omega)$ by what we have just done, Theorem 6.17 in [15] yields that $u|_{\Omega} \in \mathrm{C}^{3,\beta}(\Omega)$.

Now, to proceed by induction, suppose we have that $u \in \mathrm{C}^{k-1,\beta}$ and the metric $\mathrm{g} \in \mathrm{C}^{k,\alpha}$. Then, $f \in \mathrm{C}^{k-1,\beta}$ and the coefficients of L are $\mathrm{C}^{k-1,\alpha}$. Hence, Theorem 6.17 in [15] gives that $u|_{\Omega} \in \mathrm{C}^{k+1,\beta}(\Omega)$. That is, $\rho^{\mathrm{g}}_t \in \mathrm{C}^{k+1,\beta}(\mathcal{N}^2)$. □

7. $\mathrm{RCD}(K,N)$ spaces and singularities

In this section, we first demonstrate that the flow defined by (GM) is equal to the flow that Gigli and Mantegazza define for $\mathrm{RCD}(K,N)$ spaces in [13]. In fact, for a smooth initial metric, they verify this fact in their paper. We ensure that this is true in our more general setting.

We then consider the flow defined as (GM) on manifolds with geometric singularities away from the singular region. The correspondence we establish between this and the flow of $\mathrm{RCD}(K,N)$ defined by Gigli-Mantegazza then allows us to assert that this flow can be described by an evolving metric tensor away from the singular region for certain g_t-admissible points.

7.1. **Correspondence to the flow for** $\mathrm{RCD}(K,N)$ **spaces.** First, we recall some terminology that will be essential for the material we present here. Let $(\mathcal{X}, \mathrm{d}, \mu)$ be a compact measure metric space, and denote set of probability measures by $\mathscr{P}(\mathcal{X})$. This set can be made into a metric space under

$$W_2(\nu,\sigma)^2 = \inf\left\{\int_{\mathcal{X}\times\mathcal{X}} \mathrm{d}(x,y)^2\, d\pi : \pi \text{ is a transport map from } \nu \text{ to } \sigma\right\},$$

where by transport map, we mean that $\pi(A\times\mathcal{X}) = \nu(A)$ and $\pi(\mathcal{X}\times B) = \sigma(B)$. The metric W_2 is the *Wasserstein metric* and the space $(\mathscr{P}(\mathcal{X}), W_2)$ is the *Wasserstein space*. An important feature is that, when d is a length space, so is $(\mathscr{P}(\mathcal{X}), W_2)$ and when d is a geodesic space, then the same property holds for $(\mathscr{P}(\mathcal{X}), W_2)$.

In their paper [13], the authors demonstrate that the flow defined by (GM) for initial smooth metrics coincides with a flow which they define as a heat-flow in Wasserstein space. Namely, they demonstrate that

$$\mathrm{g}_t(\gamma_s', \gamma_s') = |\dot{\nu}_s|,$$

where $\nu_s = \rho_t^{\mathrm{g}}(\gamma_s, \cdot\,)\, d\mu$ and where $|\dot{\nu}_s|$ is the W_2 metric speed of the curve ν_s.

In the following theorem, we verify this is indeed the case when $(\mathcal{M}, \mathrm{g})$ with g rough and inducing a distance metric satisfying an $\mathrm{RCD}(K,N)$ condition. The proof is essentially the same as in the proof of Theorem 3.6 in [13], which in turn relies on the uniqueness of solutions of the continuity equation stated as Theorem 2.5 in [13], when the underlying space is a Riemannian manifold with a smooth metric. The proof of their Theorem 2.5 fails to hold in our setting as they resort to Euclidean results via the Nash embedding theorem which is unable given the low regularity of our metric.

Moreover, we note that the set $\mathcal{N}$ may not be convex with respect to g_t. Recall that two points $x, y \in \mathcal{M}$ are g_t-admissible if for every absolutely continuous curve $\gamma : I \to \mathcal{M}$ with $\gamma(0) = x$ and $\gamma(1) = y$, there is another absolutely continuous curve $\gamma' : I \to \mathcal{M}$ with $\gamma'(s) \in \mathcal{N}$ for s-a.e. for which $\ell_{\mathrm{d}_t}(\gamma') \le \ell_{\mathrm{d}_t}(\gamma)$ where

$$\ell_{\mathrm{d}_t}(\gamma) = \int_{\mathcal{M}} |\dot{\gamma}(s)|_{\mathrm{d}_t}\, ds,$$

and where $|\dot{\gamma}(s)|_{\mathrm{d}_t}$ is the metric speed of the curve computed with respect to d_t. With this terminology at hand, we present the following important theorem.

Theorem 7.1. *Let $(\mathcal{M}, \mathrm{g})$ be a smooth manifold with a rough metric and suppose that g induces a length structure such that $(\mathcal{M}, \mathrm{d}_{\mathrm{g}}, d\mu_{\mathrm{g}})$ is $\mathrm{RCD}(K,N)$. Let g_t be the flow given by Theorem 3.1 on an open subset $\varnothing \neq \mathcal{N}$. Suppose $s \mapsto \gamma_s \in \mathcal{M}$ is an absolutely continuous curve between two admissible points $x, y \in \mathcal{M}$ for which $\gamma(s) \in \mathcal{N}$ for s-a.e. Fix $t > 0$ and define*

$$\nu_s := \rho_t^{\mathrm{g}}(\gamma_s, \cdot\,) d\mu_{\mathrm{g}} = H_t(\nu_{0,\gamma_s}),$$

where H_t denotes the heat flow and $\nu_{0,\gamma_s} = \delta_{\gamma_s}$, the delta measure at γ_s. Then, $s \mapsto \nu_s$ is absolutely continuous with respect to W_2 and for s-a.e.,

$$\mathrm{g}_t(\dot{\gamma}_s, \dot{\gamma}_s) = |\dot{\nu}_s|.$$

Moreover,

$$\mathrm{d}_t(x,y)^2 = \inf_{\gamma(s)\in\mathcal{N}\ s-a.e.} \int_{\mathcal{M}} |\dot{\gamma}_s|^2_{\mathrm{g}_t}\ ds.$$

Proof. The absolute continuity of ν_s follows from absolute continuity of γ_s and the contraction property of the heat flow in spaces with curvature bounded below. From Theorem 3.1, we know that there exist a family $\psi_{t,\gamma_s,\dot{\gamma}_s} \in \mathrm{W}^{1,2}(\mathcal{M})$ solving the following equation (in the sense of distributions)

$$-\operatorname{div}_{\mathrm{g}} \rho^{\mathrm{g}}_t(\gamma_s,\cdot)\nabla\psi_{t,\gamma_s,\dot{\gamma}_s} = \mathrm{d}_x(\rho^{\mathrm{g}}_t(\gamma_s,\cdot))(\dot{\gamma}_s)$$

Now, we note that ν_s has bounded compression i.e. $\nu_s \ll d\mu_{\mathrm{g}}$ and since we assume that $(\mathcal{M}, \mathrm{d}_{\mathrm{g}}, d\mu_{\mathrm{g}})$ is an $\mathrm{RCD}(K,N)$ space, the Sobolev space $\mathrm{W}^{1,2}(\mathcal{M})$ is Hilbert. So applying Proposition 4.5 in [12], we have

$$|\dot{\nu}_s| = \|\nabla\psi_{t,\gamma_s,\dot{\gamma}_s}\|_{\mathrm{L}^2(\nu_s)},$$

which in turn means that

$$|\dot{\nu}_s|^2 = \int_M |\nabla\psi_{t,\gamma_s,\dot{\gamma}_s}|^2 d\nu_s = \mathrm{g}_t\left(\dot{\gamma}_s,\dot{\gamma}_s\right).$$

As a direct consequence, we get

$$\mathrm{d}^2_{\mathrm{g}_t}(x,y) = \inf_{\gamma}\left\{\int_0^1 |\dot{\nu}_s|^2\ ds:\ \gamma(s)\in\mathcal{N}\ s\text{-a.e. joining } x \text{ and } y\right\}$$

Notice that the right hand side the equation above is the definition of distance given by the flow (GM). So the proof is complete. □

With this theorem at hand, and on collating results we have obtained previously, we give the following proof of Theorem 1.1.

Proof of Theorem 1.1. Since we assume that $(\mathcal{M}, \mathrm{d}_{\mathrm{g}}, d\mu_{\mathrm{g}})$ is an $\mathrm{RCD}(K,N)$ space, we know from Theorem 7.3 in [2] that $\rho^{\mathrm{g}}_t \in \mathrm{C}^{0,1}(\mathcal{M}^2)$. Moreover, we assume that $\mathrm{g} \in \mathrm{C}^k(\mathcal{M}\setminus\mathcal{S})$ for $k \geq 1$, and since $\mathcal{S} \subsetneq \mathcal{M}$ is closed, $\mathcal{M}\setminus\mathcal{S}$ is open, and so we apply Theorem 6.8 to obtain that $\rho^{\mathrm{g}}_t \in \mathrm{C}^{k+1}(\mathcal{M}^2)$. By the assumptions we've made, $k+1 \geq 2$ and hence, we invoke Theorem 3.1 to obtain the conclusion. Moreover, by Theorem 7.1, we are able to assert that $\mathrm{d}_t(x,y)$ is induced by g_t for g_t-admissible points $x, y \in \mathcal{M}$. □

7.2. **Witch's hats and boxes.** In this section, we prove Corollary 3.6 and 3.7 from §3.2.

First, we note the following theorem that will make our constructions easier.

Proposition 7.2. *The gluing of two Alexandrov spaces via an isometry between their boundaries produces an Alexandrov space with the same lower curvature bound. Moreover, such a space is an* $\mathrm{RCD}(K,N)$ *space.*

Proof. The first part of the Proposition concerning the gluing of Alexandrov spaces is in [23] by Petrunin. The curvature bounds of Lott-Sturm-Villani follow from [24] by the same author. That an Alexandrov space is RCD is due to [17] by Kuawe, Machigashira, and Shioya. □

With this tool in hand, let us first consider the case of the box. Let

$$B^n = \partial\left[-\sqrt{\frac{1}{2(n+1)}}, \sqrt{\frac{1}{2(n+1)}}\,\right]^{n+1}$$

and $G : B^n \to \mathrm{S}^n \subset \mathbb{R}^{n+1}$ be the radial projection map defined by

$$G(x) = \frac{x}{|x|}.$$

We have $B^n \subset \mathbb{B}^n_0(1)$ which means that G is an expansion and hence

$$\mathrm{d}_{B^n}(x,y) \leq \mathrm{d}_{\mathrm{S}^n}(G(x),G(y)) \leq \sqrt{2(n+1)}\mathrm{d}_{B^n}(x,y)\,.$$

The second inequality follows from the fact that $\mathrm{S}^n \subset [-1,1]^{n+1}$. Putting these together, we deduce that $G^{-1} : \mathrm{S}^n \to B^n$ is Lipschitz and that the Lipschitz constant of G^{-1} satisfies $\mathbf{Lip}(G^{-1}) \leq 1$.

Immediately, by Proposition 7.2, we obtain the proof of Corollary 3.7.

Proof of Corollary 3.7. By Proposition 7.2, we obtain that B^n is an $\mathrm{RCD}(0,n)$ space. Moreover, it is easy to see that the Riemannian metric induced via G coming from the sphere is smooth on B away from the edges and corners. Thus, we can apply Theorem 1.1 to obtain that the Gigli-Mantegazza flow for d_t is given, away from edges and the corners, by a smooth metric g_t. □

Next, let us consider the Witch's hat sphere. We follow the Example 3.2 from [18] Let $\varphi : [0,\pi] \to [0,2]$ be a smooth cut-off function with

$$\varphi(r) = 0,\ \text{for } r \in \left[0, \frac{\pi}{4}\right] \ \text{ and } \ \varphi(r) = 1,\ \text{for } r \in \left[\frac{3\pi}{4}, \pi\right]$$

and such that

$$|\varphi'(r)| \leq 1/10$$

Let

$$f(r) := \varphi(r)\left(\frac{\pi - r}{\pi}\right) + (1 - \varphi(r))\sin(r)$$

Now take the metric $\mathrm{g}_{\mathrm{witch}} = dr^2 + f(r)^2\mathrm{g}_{\mathrm{S}^n}$.

The identity map $\mathrm{Id} : (\mathrm{S}^{n+1}, \mathrm{g}_{\mathrm{S}^{n+1}}) \to (\mathrm{S}^{n+1}, \mathrm{g}_{\mathrm{witch}})$ is bi-Lipschitz as a map between two metric spaces and possesses a geometric conical singularity at one pole.

Proof of Corollary 3.6. The cone is obtained by gluing the following pieces via isometry between their boundaries. Let

$$A_1 = \left[0, \frac{\pi}{4}\right] \times_f \mathrm{S}^n,\ A_2 = \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \times_f \mathrm{S}^n \text{ and } A_3 = \left[\frac{3\pi}{4}, \pi\right] \times_f \mathrm{S}^n.$$

Then, A_1 is a spherical cap with constant sectional curvature equal to 1. Hence, it obviously is an Alexandrov space. Furthermore, A_2 is a smooth warped product with bounded sectional curvature and therefore it is also Alexandrov. Lastly, A_3 is the standard cone with cross sectional diameter $< \pi$ which is known to be Alexandrov by [7]. So, by Proposition 7.2, we obtain that it is an $\mathrm{RCD}(K, N)$ space.

Moreover, since the metric $\mathrm{g}_{\mathrm{witch}}$ has a geometric conical singularity at one point, and it is smooth away from that point, by Theorem 1.1, we obtain that the Gigli-Mantegazza flow d_t is induced everywhere but at the singularity by a smooth Riemannian metric g_t. □

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