Optimal Transport and Applications

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8 A Priori Estimates and the Geometry of the Monge-Ampère Equation

after L. Caffarelli
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Abstract
We will very briefly touch on Caffarelli's regularity theory for fully nonlinear PDE and the geometry and regularity of the Monge-Ampère equation.

8.1 Introduction

The well known regularity results for small perturbation of linear equations are as follows:

(I) [**Cordes-Nienberg Type Estimates**]. Let $0 < \alpha < 1$ and u a bounded solution on B_1 of $Lu = a_{ij}D_{ij}u = f$, $|a_{ij} - \delta_{ij}| \leq \delta_0(\alpha)$ small enough, and f bounded; then,

$$||u||_{C^{1,\alpha}(B_{1/2})} \leq C \left(||u||_{L^\infty(B_1)} + ||f||_{L^\infty} \right). \tag{1}$$

(II) [**Calderon-Zygmund**]. if $f \in L^p$ for some $1 < p < \infty$ and $\delta_0(p)$ small enough, then

$$||u||_{W^{2,p}(B_{1/2})} \leq C \left(||u||_{L^\infty(B_1)} + ||f||_{L^p} \right). \tag{2}$$

(III) [**Schauder**]. If a_{ij} and f are of class C^α then,

$$||u||_{C^{2,\alpha}(B_{1/2})} \leq C \left(||u||_{L^\infty(B_1)} + ||f||_{C^\alpha} \right). \tag{3}$$

Caffarelli [1] has generalized these type of estimates to Fully nonlinear uniformly elliptic PDEs.

8.2 Fully Nonlinear Uniformly Elliptic PDE

The equation is of the form

$$F(D^2u, x) = f(x) \tag{4}$$

Uniform ellipticity in D^2 of equation 4 means that there exist λ and Λ such that for any matrix $N \in M_{n\times n}$ and any $N \in S^+$ we have

$$0 < \lambda||M|| < F(N + M, x) - F(N, x) \leq \Lambda||M|| \tag{5}$$

Definition 1 (Viscosity Solution)**.** *The continuous function u is called a C^2-viscosity solution of (4) if for any C^2-subsolution (resp. supersolution) ϕ, $u - \phi$ cannot have an interior minimum (resp. maximum).*

Let S denote the symmetric matrices then, $\beta(x)$, the oscillation of F in the variable x is given by:

$$\beta(x) = \sup_{M \in S} \frac{F(M,x) - F(M,0)}{||M||} \tag{6}$$

Caffarelli's results are as follows:

8.3 Caffrelli's Main Results

Theorem 2 ($W^{2,p}$ Regularity)**.** *Let u be a bounded viscosity solution of $F(D^2u, x) = f(x)$ in B_1 and assume that solutions ω of the Dirichlet problem*

$$f(x) = \begin{cases} F(D^2u, x) = 0 & in \;\; B_r \\ \omega = \omega_0 & in \;\; \partial B_r \end{cases}$$

satisfy he interior apriori estimate

$$||\omega||_{C^{1,1}(B_{r/2})} \leq Cr^{-2}||\omega||_{L^\infty(\partial B_r)} \tag{7}$$

Let $n < p < \infty$ and assume that $f \in L^p$ and for some $\theta = \theta(p)$ suficiently small

$$\sup_{B_1} \beta(x) \leq \theta(p) \tag{8}$$

Then $u|_{B_{1/2}}$ is in $W^{2,p}$ and

$$||u||_{W^{2,p}(B_{1/2})} \leq C \left(\sup_{\partial B_1} |u| + ||f||_{L^p} \right). \tag{9}$$

Theorem 3 ($C^{1,\alpha}$ Regularity)**.** *Assume that solutions ω to the equation*

$$F(D^2u, \omega) = 0 \tag{10}$$

in B_r satisfy the a priori estimate

$$||\omega||_{C^{1,\bar{\alpha}}(B_{r/2})} \leq Cr^{-(1+\bar{\alpha})}||\omega||_{L^\infty(B_r)} \tag{11}$$

Then for any $0 < \alpha < \bar{\alpha}$ there exists $\theta = \theta(\alpha)$ so that if

$$\fint_{B_r} \beta^n(x)dx \le \theta \tag{12}$$

and

$$\fint_{B_r} |f(x)|^n dx \le C_1 r^{(\alpha-1)n} \tag{13}$$

then any bounded solution u of

$$F(D^2u, x) = f(x) \tag{14}$$

in B_{r_0} is $C^{1,\alpha}$ at the origin. That is, there exist a linear function l such that for $r < r_0$

$$|u - l| \le C_2 r^{1+\alpha} \tag{15}$$

and

$$||l||_{C^1} \le C_3 \tag{16}$$

with

$$C_2, C_3 \le C(\alpha) r_0^{-(1+\alpha)} \sup_{B_{r_0}} |u| + C_1^{1/n} \tag{17}$$

Theorem 4 ($C^{2,\alpha}$ Regularity)**.** *Assume the existence of $C^{2,\bar{\alpha}}$ interior a priori estimates for solutions of*

$$F(D^2\omega + M, 0) = 0 \tag{18}$$

for any M satisfying

$$F(M, 0) = F(0, 0) = 0. \tag{19}$$

Then if $0 < \alpha < \bar{\alpha}$,

$$\fint_{B_r} \beta^n dx \le C r^{\alpha n}, \tag{20}$$

$$\fint_{B_r} |f(x)|^n dx \le C r^{\alpha n} \tag{21}$$

and u is a solution of $F(D^2u, x) = f(x)$, then, u is $C^{2,\alpha}$ at the origin (in the same sense as above).

These regularity results are proven by exploring Alexander-Bakelman-Pucci maximum principle and Krylov-Safanov Harnack's Inequality.

8.4 Monge-Ampère Equation

We will discuss the solutions to the Monge-Ampère equation, $\det D_{ij}u = f$ and $0 < \lambda_1 \leq f \leq \lambda_2 < \infty$ on a convex set Ω. MA equation is perhaps the most famous example of non-uniformly elliptic PDE.

8.5 Geometric Properties, Alexandrov Solutions and Localization

Solutions to the MA quation are invariant under *affine transformations* with the proper renormalization; i.e. if u is a solution and $TX = AX + B$ is an *affine transformation*, then

$$w = \frac{1}{(\det T)^{2/n}} u(TX) \tag{22}$$

is also a solution. This means that one may produce new solutions by "stretching" the graph of u in some directions and "squeezing" it in other directions (in a way that keeps the Jacobian of ∇u fixed) and hence producing singular solutions.

This fact also tells us that the estimates on the solutions are inevitably dependent on the geometry of the domain of the definition.

Definition 5 (Generalized (Alexandrov) Solutions)**.** *Let ν be a Borel measure on Ω, an open and* ***convex*** *subset of $\mathbb{R}^n$. The convex function $u \in C(\Omega)$ is a generalized solution or Alexandrov solution to the MA equation*

$$\det D^2 u = \nu \tag{23}$$

if the MA measure Mu equals ν. Mu is defined as follows:

$$Mu(E) = |\partial u(E)| \tag{24}$$

Remark 6. *For the MA equation* $\det D^2 u = f$*, we take $\nu = f\mathcal{L}$ ($\mathcal{L}$: Lebesque measure.)*

Proposition 7. *if f is continuous then every Alexandrov solution u is also a viscosity solution.*

Lemma 8. *If $u, v \in C(\bar{\Omega})$, $u|_{\partial\Omega} = v_{\partial\Omega}$ and $v \geq u$ in Ω, then,*

$$\partial v(\Omega) \subset \partial u(\Omega) \tag{25}$$

Consequences of Lemma 8:

Theorem 9 (Alexandrov Maximum Principle)**.** *$u : \Omega \to \mathbb{R}$ convex and $u|_{\partial u} = 0$ then,*

$$|u(x)|^n \leq C_n(diam\Omega)^{n-1} dist(x, \partial\Omega)|\partial\Omega| \ \ \forall x \in \Omega \tag{26}$$

Lemma 10 (Comparison Principle)**.** *Let u, v convex functions on open bounded convex Ω and $u \geq v$ on $\partial\Omega$. If $\det D^2 u \leq \det D^2 v$ (in the MA measure sense) then,*

$$u \geq v \ \ in \ \ \Omega \tag{27}$$

One key tool in studying MA equation is John's Lemma:

Lemma 11 (John's Lemma)**.** *For any open bounded convex set O, there exist an ellipsoid E such that*

$$E \subset S \subset nE \tag{28}$$

hence, there exist an invertible orientation preserving affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(S)$ is normalized i.e. $B_1 \subset T(S) \subset B_n$

One immediate consequence is the following: Let u is a strictly convex solution of MA inequality on Ω then for any $x \in \Omega' \subset\subset \Omega$ and t sufficiently small; Then, if T normalizes the section $S(x, p, t)$, then the normalization, u^* of u given by:

$$u^*(y) := (\det T)^{2/n} \left(u(T^{-1}(y)) - u(x) - p.(T^{-1}(y) - x) - t \right) \tag{29}$$

solves the MA inequality on $T(S(x, p, t))$ with boundary condition $u^*\Big|_{\partial T(S(x,p,t))} = 0$.

Lemma 12. *Let Ω^* be a normalized open convex set i.e. $B_1 \subset \Omega^* \subset B_n$ and let u^* solve*

$$\lambda_1 \leq \det D^2 u \leq \lambda_2 \quad u^*|_{\partial\Omega^*} = 0 \tag{30}$$

(we call u^* a normalized solution) then, there constants c_1, c_2 depending on λ_1, λ_2 such that

$$0 < c_1 \leq \left| \inf_{\Omega^*} u^* \right| \leq c_2 \tag{31}$$

Proof. Apply Lemma 10 to $\omega_1 = \lambda_1(|x|^2 - 1)/2$ and $\omega_2 = \lambda_2(|x|^2 - n^2)/2$ □

Another important result is the localization Theorem which will be used in the proof of compactness theorem and strict convexity of solutions.

Theorem 13 (Localization [2])**.** *Let u be a solution of* ***MA inequality***

inside a convex set Ω, and let $l(x)$ is a supporting sloe to u. If the convex set

$$W = \{u(x) = l(x)\} \tag{32}$$

contains more than one point (hence not strictly convex) then it can not have an extremal point in Ω (i.e. this set has to exit the domain of the definition).

As a consequence of Lemmas 13, we get the compactness for normalized solutions. [1]

The proof of regularity results uses the properties of the sections of the solutions i.e. the sets

$$S(x,p,t) := \{y \in \Omega : u(y) \leq u(x)+p.(y-x)+t\} \ \ where, \ \ p \in \partial u(x) \ \ and \ \ t \geq 0. \tag{33}$$

the modulus of convexity is defined to be

$$\omega(x,u,t) := \sup_{p\in\partial u(x)} diam S(x,p,t) \tag{34}$$

and

$$\omega_{\Omega'} = \sup_{x\in\Omega'} \omega(x,u,t) \ \ \Omega' \subset\subset \Omega \tag{35}$$

8.6 $C^{1,\alpha}$ Regularity

It is enough to prove the $C^{1,\alpha}$ regularity of renormalized solutions u with $\inf_\Omega u = u(x_0)$ and $u|_{\partial\Omega} = 0$. Let $C_\beta \subset \mathbb{R}^{n-1}$ be the cone with vertex $(x_0, u(x_0))$ and base $\{u = (1-\beta)u(x_0)\}$. Suppose C_α is the graph of h_α.

Using the compactness result, one can find a universal δ for which

$$h_{1/2} \leq (1-\delta)h_1 \tag{36}$$

After renormalizing the level surface $\{u = 2^{-k}\}$ and iteration, we get:

$$h_{2^{-k}} \leq (1-\delta)^k h_1 \tag{37}$$

Since u is Lipschitz we have $h_1(x) \le C|x-x_0|+u(x_0)$. Letting $2^{-\alpha'} = 1-\delta$ we get

$$h_{2^{-k}}(x) \le C(2^{-k})^{\alpha}|x| \le C2^{-k} \tag{38}$$

for $|x| \le (2^{-k})^{(1-\alpha)}$

By the comparison Lemma 10, we have $u(x) \le h_{2^{-k}}(x)$ as long as $h_{2^k}(x) \le (1-2^{-k})u(x_0)$. This means that if for every x, we pick k that satisfies

$$(\frac{-C}{u(x_0)})2^{-(k+1)(1-\alpha')} \le |x-x_0| \le (\frac{-C}{u(x_0)})2^{-k(1-\alpha')} \tag{39}$$

then,

$$h_{2^k}(x) \le (1-2^{-k})u(x_0) \tag{40}$$

And direct computation gives:

$$u(x)-u(x_0) \le C|x-x_0|^{1+\alpha} \ \ where, \ \ \alpha = \frac{\alpha'}{1-\alpha'} \tag{41}$$

This shows that for all supporting planes l_{x_0}, we have:

$$\sup_{B(x_0,r)} |u(x) - l_{x_0}(x)| \le Cr^{1+\alpha} \tag{42}$$

and this will imply that u is $C^{1,\alpha}$.

8.7 Sobolev Regularity

Theorem 14 (Caffarelli [1]). *Let u be a convex viscosity solution of the MA equation on a normalized convex set Ω and $u|_{\partial\Omega} = 0$ then,*

(I) *$\forall p < \infty, \exists \epsilon = \epsilon(p)$ s.t. if*

$$|f-1| \le \epsilon \tag{43}$$

then,

$$u \in W^{2,p}\left(B_{1/2}\right) \tag{44}$$

and

$$||u||_{W^{2,p}\left(B_{1/2}\right)} \le C(\epsilon) \tag{45}$$

(II) *If $f > 0$ and is continuous, then $u \in W^{2,p}\left(B_{1/2}\right)$ for any $p < \infty$ and*

$$||u||_{W^{2,p}\left(B_{1/2}\right)} \le C(p,\sigma) \tag{46}$$

where σ is the modulus of continuity of f.

A consequence is the following theorem:

Theorem 15. $f \in C^\alpha \implies u \in C^{2,\alpha}$

Main Ideas of the Proof:

Lets consider a particular case: $1 \leq \det D_{ij}u \leq 1 + \epsilon(p)$ and we want to prove that $||u||_{W^{2,p}\left(B_{1/2}\right)} \leq C(p)$.

Step 1 Take the section $S_{\mu,L} = \{u - L \leq min(u - L) + \mu\}$ and normalize it by T_μ. Then approximate (using an approximation lemma as in [1]) the normalization of $u - L$ by solutions of $\det D_{ij}\omega = 1$. Notice that ω is $C^{2,\alpha}$

Step 2 Iterating this approximation at diadic levels $\mu = 2^{-k}$, one can show that

$$T_m u = D_\mu \tilde{T}_m u \tag{47}$$

where $D_m u = \left(\frac{1}{2\mu}\right)^{1/2}$ Id is a dilation and and $\tilde{T}_m u$ is a transformation of norm

$$||\tilde{T}_\mu|| \; , \; ||\tilde{T}_m u^{-1}|| \leq \mu^{-\sigma} \tag{48}$$

with $\sigma = \sigma(\epsilon)$ is as small as we want.

So far we have a normalized solution u on $T_\mu(S_{\mu,L})$ with the following properties:

(a) $1 \leq \det D_{ij}u \leq 1+\epsilon$. **(b)** $\{u = 1\}$ is trapped between B_1 and B_n. **(c)** u is ϵ away from the $C^{2,\alpha}$ approximation function ω that solves $\det D_{ij}\omega = 1$ and $\{\omega = 0\} = \{u = 0\}$

Step 3

Lemma 16. *Let $\Gamma(u - \frac{1}{2}\omega)$ be the convex envelope of $u - \frac{1}{2}\omega$, then, the contact set $C = \{\Gamma(u - \frac{1}{2}\omega) = u - \frac{1}{2}\omega\}$ satisfies:*

$$\frac{|B_{1/2} \cap C|}{|B_{1/2}|} \geq 1 - C\epsilon^{1/2} \tag{49}$$

in other words, the contact points cover as large a portion of $B_{1/2}$ as we want.

Corollary 17. *At any contact point x_0, there exist a plane L_{x_0} such that in all of Ω*

$$L_{x_0}(x) \leq (u - \frac{1}{2}\omega)(x) \;\; and \;\; L_{x_0}(x_0) = (u - \frac{1}{2}\omega)(x_0) \tag{50}$$

which means that for any contact point x_0, u has a tangent paraboloid by below of the form

$$L_{x_0} + \frac{1}{N}|x - x_0|^2 \tag{51}$$

Remark 18. *If u has a tangent paraboloid by below $u \geq \frac{1}{\lambda}|x|^2$ then u has a tangent paraboloid by above $u \leq \lambda^{n-1}|x|^2$ because one can see that a paraboloid from below puts a uniform bound $||\tilde{T}_\mu|| \leq \lambda$ then since $\det \tilde{T}_m u = 1$, we also get a bound by below.*

Step 4 having controlled tangent paraboloids from above and below $\implies$ $W^{2,p}$ estimates.

References

[1] Caffarelli, L. A, *A priori estimates and the geometry of the Monge Ampère equation.* IAS/Park City Math. Ser. **2** (1996), no. 1, 5–63;

[2] Caffarelli, L. A, *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity.* Ann. of Math. **131** (1990), no. 1, 129–134;