# Optimal Transport and Applications 

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Organizers:<br>Nestor Guillen<br>University of California, Los Angeles, USA<br>Dimitri Shlyakhtenko<br>University of California, Los Angeles, USA<br>Christoph Thiele<br>University of California, Los Angeles, USA / University of Bonn, Germany

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# 8 A Priori Estimates and the Geometry of the Monge-Ampère Equation 

after L. Caffarelli<br>A summary written by Sajjad Lakzian


#### Abstract

We will very briefly touch on Caffarelli's regularity theory for fully nonlinear PDE and the geometry and regularity of the Monge-Ampère equation.


### 8.1 Introduction

The well known regularity results for small perturbation of linear equations are as follows:
(I) [Cordes-Nienberg Type Estimates]. Let $0<\alpha<1$ and $u$ a bounded solution on $B_{1}$ of $L u=a_{i j} D_{i j} u=f,\left|a_{i j}-\delta_{i j}\right| \leq \delta_{0}(\alpha)$ small enough, and $f$ bounded; then,

$$
\begin{equation*}
\|u\|_{C^{1, \alpha}\left(B_{1 / 2}\right)} \leq C\left(\|u\|_{L^{\infty}\left(B_{1}\right)}+\|f\|_{L^{\infty}}\right) \tag{1}
\end{equation*}
$$

(II) [Calderon-Zygmund]. if $f \in L^{p}$ for some $1<p<\infty$ and $\delta_{0}(p)$ small enough, then

$$
\begin{equation*}
\|u\|_{W^{2, p}\left(B_{1 / 2}\right)} \leq C\left(\|u\|_{L^{\infty}\left(B_{1}\right)}+\|f\|_{L^{p}}\right) . \tag{2}
\end{equation*}
$$

(III) [Schauder]. If $a_{i j}$ and $f$ are of class $C^{\alpha}$ then,

$$
\begin{equation*}
\|u\|_{C^{2, \alpha}\left(B_{1 / 2}\right)} \leq C\left(\|u\|_{L^{\infty}\left(B_{1}\right)}+\|f\|_{C^{\alpha}}\right) . \tag{3}
\end{equation*}
$$

Caffarelli [1] has generalized these type of estimates to Fully nonlinear uniformly elliptic PDEs.

### 8.2 Fully Nonlinear Uniformly Elliptic PDE

The equation is of the form

$$
\begin{equation*}
F\left(D^{2} u, x\right)=f(x) \tag{4}
\end{equation*}
$$

Uniform ellipticity in $D^{2}$ of equation 4 means that there exist $\lambda$ and $\Lambda$ such that for any matrix $N \in M_{n \times n}$ and any $N \in S^{+}$we have

$$
\begin{equation*}
0<\lambda\|M\|<F(N+M, x)-F(N, x) \leq \Lambda\|M\| \tag{5}
\end{equation*}
$$

Definition 1 (Viscosity Solution). The continuous function $u$ is called a $C^{2}$-viscosity solution of (4) if for any $C^{2}-$ subsolution (resp. supersolution) $\phi, u-\phi$ cannot have an interior minimum (resp. maximum).

Let $S$ denote the symmetric matrices then, $\beta(x)$, the oscillation of $F$ in the variable $x$ is given by:

$$
\begin{equation*}
\beta(x)=\sup _{M \in S} \frac{F(M, x)-F(M, 0)}{\|M\|} \tag{6}
\end{equation*}
$$

Caffarelli's results are as follows:

### 8.3 Caffrelli's Main Results

Theorem 2 ( $W^{2, p}$ Regularity). Let $u$ be a bounded viscosity solution of $F\left(D^{2} u, x\right)=f(x)$ in $B_{1}$ and assume that solutions $\omega$ of the Dirichlet problem

$$
f(x)=\left\{\begin{array}{lr}
F\left(D^{2} u, x\right)=0 & \text { in } B_{r} \\
\omega=\omega_{0} & \text { in } \partial B_{r}
\end{array}\right.
$$

satisfy he interior apriori estimate

$$
\begin{equation*}
\|\omega\|_{C^{1,1}\left(B_{r / 2}\right)} \leq C r^{-2}\|\omega\|_{L^{\infty}\left(\partial B_{r}\right)} \tag{7}
\end{equation*}
$$

Let $n<p<\infty$ and assume that $f \in L^{p}$ and for some $\theta=\theta(p)$ suficiently small

$$
\begin{equation*}
\sup _{B_{1}} \beta(x) \leq \theta(p) \tag{8}
\end{equation*}
$$

Then $\left.u\right|_{B_{1 / 2}}$ is in $W^{2, p}$ and

$$
\begin{equation*}
\|u\|_{W^{2, p}\left(B_{1 / 2}\right)} \leq C\left(\sup _{\partial B_{1}}|u|+\|f\|_{L^{p}}\right) \tag{9}
\end{equation*}
$$

Theorem 3 ( $C^{1, \alpha}$ Regularity). Assume that solutions $\omega$ to the equation

$$
\begin{equation*}
F\left(D^{2} u, \omega\right)=0 \tag{10}
\end{equation*}
$$

in $B_{r}$ satisfy the a priori estimate

$$
\begin{equation*}
\|\omega\|_{C^{1, \bar{\alpha}}\left(B_{r / 2}\right)} \leq C r^{-(1+\bar{\alpha})}\|\omega\|_{L^{\infty}\left(B_{r}\right)} \tag{11}
\end{equation*}
$$

Then for any $0<\alpha<\bar{\alpha}$ there exists $\theta=\theta(\alpha)$ so that if

$$
\begin{equation*}
f_{B_{r}} \beta^{n}(x) d x \leq \theta \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{B_{r}}|f(x)|^{n} d x \leq C_{1} r^{(\alpha-1) n} \tag{13}
\end{equation*}
$$

then any bounded solution $u$ of

$$
\begin{equation*}
F\left(D^{2} u, x\right)=f(x) \tag{14}
\end{equation*}
$$

in $B_{r_{0}}$ is $C^{1, \alpha}$ at the origin. That is, there exist a linear function $l$ such that for $r<r_{0}$

$$
\begin{equation*}
|u-l| \leq C_{2} r^{1+\alpha} \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\|l\|_{C^{1}} \leq C_{3} \tag{16}
\end{equation*}
$$

with

$$
\begin{equation*}
C_{2}, C_{3} \leq C(\alpha) r_{0}^{-(1+\alpha)} \sup _{B_{r_{0}}}|u|+C_{1}^{1 / n} \tag{17}
\end{equation*}
$$

Theorem 4 ( $C^{2, \alpha}$ Regularity). Assume the existence of $C^{2, \bar{\alpha}}$ interior a priori estimates for solutions of

$$
\begin{equation*}
F\left(D^{2} \omega+M, 0\right)=0 \tag{18}
\end{equation*}
$$

for any $M$ satisfying

$$
\begin{equation*}
F(M, 0)=F(0,0)=0 \tag{19}
\end{equation*}
$$

Then if $0<\alpha<\bar{\alpha}$,

$$
\begin{gather*}
f_{B_{r}} \beta^{n} d x \leq C r^{\alpha n}  \tag{20}\\
f_{B_{r}}|f(x)|^{n} d x \leq C r^{\alpha n} \tag{21}
\end{gather*}
$$

and $u$ is a solution of $F\left(D^{2} u, x\right)=f(x)$, then, $u$ is $C^{2, \alpha}$ at the origin (in the same sense as above).

These regularity results are proven by exploring Alexander-BakelmanPucci maximum principle and Krylov-Safanov Harnack's Inequality.

### 8.4 Monge-Ampère Equation

We will discuss the solutions to the Monge-Ampère equation, $\operatorname{det} D_{i j} u=f$ and $0<\lambda_{1} \leq f \leq \lambda_{2}<\infty$ on a convex set $\Omega$. MA equation is perhaps the most famous example of non-uniformly elliptic PDE.

### 8.5 Geometric Properties, Alexandrov Solutions and Localization

Solutions to the MA quation are invariant under affine transformations with the proper renormalization; i.e. if $u$ is a solution and $T X=A X+B$ is an affine transformation, then

$$
\begin{equation*}
w=\frac{1}{(\operatorname{det} T)^{2 / n}} u(T X) \tag{22}
\end{equation*}
$$

is also a solution. This means that one may produce new solutions by "stretching" the graph of $u$ in some directions and "squeezing" it in other directions (in a way that keeps the Jacobian of $\nabla u$ fixed) and hence producing singular solutions.

This fact also tells us that the estimates on the solutions are inevitably dependent on the geometry of the domain of the definition.

Definition 5 (Generalized (Alexandrov) Solutions). Let $\nu$ be a Borel measure on $\Omega$, an open and convex subset of $\mathbb{R}^{n}$. The convex function $u \in C(\Omega)$ is a generalized solution or Alexandrov solution to the MA equation

$$
\begin{equation*}
\operatorname{det} D^{2} u=\nu \tag{23}
\end{equation*}
$$

if the $M A$ measure $M u$ equals $\nu . M u$ is defined as follows:

$$
\begin{equation*}
M u(E)=|\partial u(E)| \tag{24}
\end{equation*}
$$

Remark 6. For the $M A$ equation $\operatorname{det} D^{2} u=f$, we take $\nu=f \mathcal{L}(\mathcal{L}$ : Lebesque measure.)

Proposition 7. if $f$ is continuous then every Alexandrov solution $u$ is also a viscosity solution.

Lemma 8. If $u, v \in C(\bar{\Omega}),\left.u\right|_{\partial \Omega}=v_{\partial \Omega}$ and $v \geq u$ in $\Omega$, then,

$$
\begin{equation*}
\partial v(\Omega) \subset \partial u(\Omega) \tag{25}
\end{equation*}
$$

Consequences of Lemma 8:
Theorem 9 (Alexandrov Maximum Principle). $u: \Omega \rightarrow \mathbb{R}$ convex and $\left.u\right|_{\partial u}=0$ then,

$$
\begin{equation*}
|u(x)|^{n} \leq C_{n}(\operatorname{diam} \Omega)^{n-1} \operatorname{dist}(x, \partial \Omega)|\partial \Omega| \quad \forall x \in \Omega \tag{26}
\end{equation*}
$$

Lemma 10 (Comparison Principle). Let $u$, $v$ convex functions on open bounded convex $\Omega$ and $u \geq v$ on $\partial \Omega$. If $\operatorname{det} D^{2} u \leq \operatorname{det} D^{2} v$ (in the $M A$ measure sense) then,

$$
\begin{equation*}
u \geq v \text { in } \Omega \tag{27}
\end{equation*}
$$

One key tool in studying MA equation is John's Lemma:
Lemma 11 (John's Lemma). For any open bounded convex set $O$, there exist an ellipsoid $E$ such that

$$
\begin{equation*}
E \subset S \subset n E \tag{28}
\end{equation*}
$$

hence, there exist an invertible orientation preserving affine transformation $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ such that $T(S)$ is normalized i.e. $B_{1} \subset T(S) \subset B_{n}$

One immediate consequence is the following: Let $u$ is a strictly convex solution of MA inequality on $\Omega$ then for any $x \in \Omega^{\prime} \subset \subset \Omega$ and $t$ sufficiently small; Then, if $T$ normalizes the section $S(x, p, t)$, then the normalization, $u^{*}$ of $u$ given by:

$$
\begin{equation*}
u^{*}(y):=(\operatorname{det} T)^{2 / n}\left(u\left(T^{-1}(y)\right)-u(x)-p \cdot\left(T^{-1}(y)-x\right)-t\right) \tag{29}
\end{equation*}
$$ 0.

solves the MA inequality on $T(S(x, p, t))$ with boundary condition $\left.u^{*}\right|_{\partial T(S(x, p, t))}=$

Lemma 12. Let $\Omega^{*}$ be a normalized open convex set i.e. $B_{1} \subset \Omega^{*} \subset B_{n}$ and let $u^{*}$ solve

$$
\begin{equation*}
\lambda_{1} \leq \operatorname{det} D^{2} u \leq\left.\lambda_{2} \quad u^{*}\right|_{\partial \Omega^{*}}=0 \tag{30}
\end{equation*}
$$

(we call $u^{*}$ a normalized solution) then, there constants $c_{1}, c_{2}$ depending on $\lambda_{1}, \lambda_{2}$ such that

$$
\begin{equation*}
0<c_{1} \leq\left|\inf _{\Omega^{*}} u^{*}\right| \leq c_{2} \tag{31}
\end{equation*}
$$

Proof. Apply Lemma 10 to $\omega_{1}=\lambda_{1}\left(|x|^{2}-1\right) / 2$ and $\omega_{2}=\lambda_{2}\left(|x|^{2}-n^{2}\right) / 2$

Another important result is the localization Theorem which will be used in the proof of compactness theorem and strict convexity of solutions.

Theorem 13 (Localization [2]). Let u be a solution of MA inequality inside a convex set $\Omega$, and let $l(x)$ is a supporting sloe to $u$. If the convex set

$$
\begin{equation*}
W=\{u(x)=l(x)\} \tag{32}
\end{equation*}
$$

contains more than one point (hence not strictly convex) then it can not have an extremal point in $\Omega$ (i.e. this set has to exit the domain of the definition ).

As a consequence of Lemmas 13, we get the compactness for normalized solutions. [1]

The proof of regularity results uses the properties of the sections of the solutions i.e. the sets
$S(x, p, t):=\{y \in \Omega: u(y) \leq u(x)+p .(y-x)+t\}$ where, $p \in \partial u(x)$ and $t \geq 0$.
the modulus of convexity is defined to be

$$
\begin{equation*}
\omega(x, u, t):=\sup _{p \in \partial u(x)} \operatorname{diamS}(x, p, t) \tag{34}
\end{equation*}
$$

and

$$
\begin{equation*}
\omega_{\Omega^{\prime}}=\sup _{x \in \Omega^{\prime}} \omega(x, u, t) \quad \Omega^{\prime} \subset \subset \Omega \tag{35}
\end{equation*}
$$

## 8.6 $C^{1, \alpha}$ Regularity

It is enough to prove the $C^{1, \alpha}$ regularity of renormalized solutions $u$ with $\inf _{\Omega} u=u\left(x_{0}\right)$ and $\left.u\right|_{\partial \Omega}=0$. Let $C_{\beta} \subset \mathbb{R}^{n-1}$ be the cone with vertex $\left(x_{0}, u\left(x_{0}\right)\right)$ and base $\left\{u=(1-\beta) u\left(x_{0}\right)\right\}$. Suppose $C_{\alpha}$ is the graph of $h_{\alpha}$.

Using the compactness result, one can find a universal $\delta$ for which

$$
\begin{equation*}
h_{1 / 2} \leq(1-\delta) h_{1} \tag{36}
\end{equation*}
$$

After renormalizing the level surface $\left\{u=2^{-k}\right\}$ and iteration, we get:

$$
\begin{equation*}
h_{2^{-k}} \leq(1-\delta)^{k} h_{1} \tag{37}
\end{equation*}
$$

Since $u$ is Lipschitz we have $h_{1}(x) \leq C\left|x-x_{0}\right|+u\left(x_{0}\right)$. Letting $2^{-\alpha^{\prime}}=1-\delta$ we get

$$
\begin{equation*}
h_{2^{-k}}(x) \leq C\left(2^{-k}\right)^{\alpha}|x| \leq C 2^{-k} \tag{38}
\end{equation*}
$$

for $|x| \leq\left(2^{-k}\right)^{(1-\alpha)}$
By the comparison Lemma 10, we have $u(x) \leq h_{2^{-k}}(x)$ as long as $h_{2^{k}}(x) \leq$ $\left(1-2^{-k}\right) u\left(x_{0}\right)$. This means that if for every $x$, we pick $k$ that satisfies

$$
\begin{equation*}
\left(\frac{-C}{u\left(x_{0}\right)}\right) 2^{-(k+1)\left(1-\alpha^{\prime}\right)} \leq\left|x-x_{0}\right| \leq\left(\frac{-C}{u\left(x_{0}\right)}\right) 2^{-k\left(1-\alpha^{\prime}\right)} \tag{39}
\end{equation*}
$$

then,

$$
\begin{equation*}
h_{2^{k}}(x) \leq\left(1-2^{-k}\right) u\left(x_{0}\right) \tag{40}
\end{equation*}
$$

And direct computation gives:

$$
\begin{equation*}
u(x)-u\left(x_{0}\right) \leq C\left|x-x_{0}\right|^{1+\alpha} \text { where, } \quad \alpha=\frac{\alpha^{\prime}}{1-\alpha^{\prime}} \tag{41}
\end{equation*}
$$

This shows that for all supporting planes $l_{x_{0}}$, we have:

$$
\begin{equation*}
\sup _{B\left(x_{0}, r\right)}\left|u(x)-l_{x_{0}}(x)\right| \leq C r^{1+\alpha} \tag{42}
\end{equation*}
$$

and this will imply that $u$ is $C^{1, \alpha}$.

### 8.7 Sobolev Regularity

Theorem 14 (Caffarelli [1]). Let u be a convex viscosity solution of the MA equation on a normalized convex set $\Omega$ and $\left.u\right|_{\partial \Omega}=0$ then,
(I) $\forall p<\infty, \exists \epsilon=\epsilon(p)$ s.t. if

$$
\begin{equation*}
|f-1| \leq \epsilon \tag{43}
\end{equation*}
$$

then,

$$
\begin{equation*}
u \in W^{2, p}\left(B_{1 / 2}\right) \tag{44}
\end{equation*}
$$

and

$$
\begin{equation*}
\|u\|_{W^{2, p}\left(B_{1 / 2}\right)} \leq C(\epsilon) \tag{45}
\end{equation*}
$$

(II) If $f>0$ and is continuous, then $u \in W^{2, p}\left(B_{1 / 2}\right)$ for any $p<\infty$ and

$$
\begin{equation*}
\|u\|_{W^{2, p}\left(B_{1 / 2}\right)} \leq C(p, \sigma) \tag{46}
\end{equation*}
$$

where $\sigma$ is the modulus of continuity of $f$.

A consequence is the following theorem:
Theorem 15. $f \in C^{\alpha} \Longrightarrow u \in C^{2, \alpha}$

## Main Ideas of the Proof:

Lets consider a particular case: $1 \leq \operatorname{det} D_{i j} u \leq 1+\epsilon(p)$ and we want to prove that $\|u\|_{W^{2, p}\left(B_{1 / 2}\right)} \leq C(p)$.

Step 1 Take the section $S_{\mu, L}=\{u-L \leq \min (u-L)+\mu\}$ and normalize it by $T_{\mu}$. Then approximate (using an approximation lemma as in [1] ) the normalization of $u-L$ by solutions of $\operatorname{det} D_{i j} \omega=1$. Notice that $\omega$ is $C^{2, \alpha}$

Step 2 Iterating this approximation at diadic levels $\mu=2^{-k}$, one can show that

$$
\begin{equation*}
T_{m} u=D_{\mu} \tilde{T}_{m} u \tag{47}
\end{equation*}
$$

where $D_{m} u=\left(\frac{1}{2 \mu}\right)^{1 / 2}$ Id is a dilation and and $\tilde{T}_{m} u$ is a transformation of norm

$$
\begin{equation*}
\left\|\tilde{T}_{\mu}\right\|,\left\|\tilde{T}_{m} u^{-1}\right\| \leq \mu^{-\sigma} \tag{48}
\end{equation*}
$$

with $\sigma=\sigma(\epsilon)$ is as small as we want.
So far we have a normalized solution $u$ on $T_{\mu}\left(S_{\mu, L}\right)$ with the following properties:
(a) $1 \leq \operatorname{det} D_{i j} u \leq 1+\epsilon$. (b) $\{u=1\}$ is trapped between $B_{1}$ and $B_{n}$. (c) $u$ is $\epsilon$ away from the $C^{2, \alpha}$ approximation function $\omega$ that solves $\operatorname{det} D_{i j} \omega=1$ and $\{\omega=0\}=\{u=0\}$

Step 3
Lemma 16. Let $\Gamma\left(u-\frac{1}{2} \omega\right)$ be the convex envelope of $u-\frac{1}{2} \omega$, then, the contact set $C=\left\{\Gamma\left(u-\frac{1}{2} \omega\right)=u-\frac{1}{2} \omega\right\}$ satisfies:

$$
\begin{equation*}
\frac{\left|B_{1 / 2} \cap C\right|}{\left|B_{1 / 2}\right|} \geq 1-C \epsilon^{1 / 2} \tag{49}
\end{equation*}
$$

in other words, the contact points cover as large a portion of $B_{1 / 2}$ as we want.
Corollary 17. At any contact point $x_{0}$, there exist a plane $L_{x_{0}}$ such that in all of $\Omega$

$$
\begin{equation*}
L_{x_{0}}(x) \leq\left(u-\frac{1}{2} \omega\right)(x) \text { and } L_{x_{0}}\left(x_{0}\right)=\left(u-\frac{1}{2} \omega\right)\left(x_{0}\right) \tag{50}
\end{equation*}
$$

which means that for any contact point $x_{0}$, $u$ has a tangent paraboloid by below of the form

$$
\begin{equation*}
L_{x_{0}}+\frac{1}{N}\left|x-x_{0}\right|^{2} \tag{51}
\end{equation*}
$$

Remark 18. If $u$ has a tangent paraboloid by below $u \geq \frac{1}{\lambda}|x|^{2}$ then $u$ has a tangent paraboloid by above $u \leq \lambda^{n-1}|x|^{2}$ because one can see that a paraboloid from below puts a uniform bound $\left\|\tilde{T}_{\mu}\right\| \leq \lambda$ then since $\operatorname{det} \tilde{T}_{m} u=1$, we also get a bound by below.

Step 4 having controlled tangent paraboloids from above and below $\Longrightarrow$ $W^{2, p}$ estimates.

## References

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